

## A sticky business: the status of the conjectured viscosity/entropy density bound

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**Aleksey Cherman, Thomas D. Cohen and Paul M. Hohler**

*Department of Physics, University of Maryland,  
College Park, MD 20742-4111, U.S.A.*

*E-mail: [alekseyc@physics.umd.com](mailto:alekseyc@physics.umd.com), [cohen@physics.umd.edu](mailto:cohen@physics.umd.edu),  
[pmhohler@physics.umd.edu](mailto:pmhohler@physics.umd.edu)*

**ABSTRACT:** There have been a number of forms of a conjecture that there is a universal lower bound on the ratio,  $\eta/s$ , of the shear viscosity,  $\eta$ , to entropy density,  $s$ , with several different domains of validity. We examine the various forms of the conjecture. We argue that a number of variants of the conjecture are not viable due to the existence of theoretically consistent counterexamples. We also note that much of the evidence in favor of a bound does not apply to the variants which have not yet been ruled out.

**KEYWORDS:**  $1/N$  Expansion, AdS-CFT Correspondence, QCD.

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## 1. Introduction

Kovtun, Son, and Starinets (KSS) have proposed a conjecture that there is a universal bound for the ratio of shear viscosity,  $\eta$ , to entropy density,  $s$ , [1]:

$$\frac{\eta}{s} \geq \frac{\hbar}{k_B} \frac{1}{4\pi}, \quad (1.1)$$

where  $\hbar$  and  $k_B$  are Plank's constant and Boltzmann's constant, respectively. (For the remainder of this paper we will use units with  $\hbar = 1$  and  $k_B = 1$ .) KSS found that eq. (1.1) is saturated by certain strongly coupled field theories which have a super-gravity dual [1], and conjectured that  $\eta/s$  has a universal lower limit. Physically interesting and accessible fluids, such as water, liquid nitrogen, and helium-4 satisfy the bound [2]. The bound appears to be well justified for the class of field theories originally considered by KSS [1], but it is not obvious from first principles that it should apply more universally (hence its status as a conjecture).

The original form of the KSS conjecture states that the bound should be universal and apply to *all* fluids, including non-relativistic fluids [1]. Yet even such an all-encompassing statement includes ambiguities. It is not clear what one might mean by “all fluids” in such a context. Is the conjecture limited to physically realizable systems, or is it equally applicable to theoretical fluids which can be constructed in some given class of theories? If so, in which class of theories does the bound hold? Is a “fluid” required to be absolutely stable, or can the fluid be metastable? Are the number of species of particle that compose the fluid limited? Perhaps due to questions such as these, a number of variants of the conjecture with various proposed domains of validity were subsequently proposed by KSS. These include variants which stipulate that the bound is valid for “all relativistic quantum field theories at finite temperature and zero chemical potential” [2], for at least a “single component nonrelativistic gas of particles with either spin zero or spin 1/2” [2], or for “all systems which can be obtained from a sensible relativistic quantum field theory by turning on temperatures and chemical potentials” [3]. While some of these variants appear quite similar at first glance, they actually have quite different regimes of validity.

If the bound could be shown to be correct in any of its proposed forms, or indeed in some readily specifiable alternative form, it would represent a truly major advance in our understanding of quantum many-body physics. Indeed, even as a conjecture it has been invoked in discussing systems as diverse as ultra-cold gases of trapped atoms [4] and the quark-gluon plasma (QGP) [5]. Since KSS first conjectured their bound, the ratio of shear viscosity to entropy density has been investigated in a variety of systems, [6–14]. The smallest reported measurement of  $\eta/s$  has been associated with the QGP at RHIC [5]. (A more recent analysis of the data from RHIC may actually be consistent with a violation of the proposed bound [15].) Since the  $\eta/s$  bound may (or may not) have a rather extensive scope, it is important to understand in which types of systems one should expect the bound to hold.

As will be discussed in some detail below, the conjectured domains of validity of the conjecture differ radically from form to form. Moreover, apparently innocuous changes in the formulation of the variants of the conjecture can radically alter the systems for which they apply. Accordingly, it is important in dealing with this subject to clarify the precise nature of the various forms of the conjecture and, in particular, to which physical systems they might apply.

The outline of this paper is as follows. In section 2, we begin with a brief discussion of evidence in favor of the KSS bound in any of its forms. In section 3, we classify a set of possible domains of applicability for which eq. (1.1) might hold. The various forms of

the conjecture proposed by KSS will form a subset of these. Having delineated the various forms, we critically examine the physical systems for which these variants actually apply. In sections 4, 5, and 6, we address the key issue of the evidence that any particular variation of the conjecture might be valid. A natural question in this context is whether one can construct a theoretical counterexample to a particular variant. In these sections, we will present counterexamples to a number of variants of the conjecture. In this context, we discuss in section 6.3 a subtle issue raised in ref. [16] regarding the interplay of thermodynamic and hydrodynamic limits for the counter-example in section 6 (a heavy meson system based on a UV-complete quantum field theory). As will be seen, while ref. [16] raises a profound issue, ultimately, it does not invalidate the counterexample.

In these sections we also point out that much of the evidence which seems to support the conjecture in some general way is applicable only to variants of the conjecture which have been ruled out by the counterexamples. Thus, our ultimate conclusion is that the evidence for the conjecture in any of its forms is rather weak. If the bound is correct, it appears that this would have to be due to some deep physics (for instance, due to some aspects of quantum gravity as suggested in ref. [17], or the string- or M- theory underlying the field theories used to describe nature) beyond the frameworks of quantum mechanics and quantum field theory.

We summarize our conclusions in section 7. We relegate a number of the computational details to various appendices.

## 2. Evidence for the KSS bound

Before we begin, it is useful to briefly review the arguments of KSS that have led to their proposed bound. The argument makes use of the AdS/CFT duality from string theory [18–21]. It is argued that in higher dimensional gravity theories, black branes (higher-dimensional analogs of black holes) have finite temperature field theory duals (specifically,  $\mathcal{N} = 4$  supersymmetric Yang-Mills theories at large  $N_c$  and infinite 't Hooft coupling  $g^2 N_c$ ) that possess hydrodynamic properties such as viscosity. These hydrodynamic properties can be related to gravitational properties of the black branes, and the correspondence can be used to compute transport properties [1]. Using these methods the ratio  $\eta/s$  can be computed. A number of theories in this class have been studied in the large  $N_c$  limit at infinite 't Hooft coupling. All of them have saturated the inequality of eq. (1.1) [1]. A general argument has been given that all theories in this class at large  $N_c$  and infinite 't Hooft coupling must saturate the bound [2]. Moreover, one generally expects that as one weakens the coupling of an interacting system, the viscosity should increase. One might, therefore, expect that as the 't Hooft coupling is decreased from infinity, the ratio  $\eta/s$  should increase. This has been seen in an explicit calculation for the first correction due to finite 't Hooft coupling for one particular theory [22]. Thus, it seems quite plausible that  $\eta/s$  is bounded as in eq. (1.1), at least for those large  $N_c$  field theories which have super-gravity holographic duals.

The interesting question is whether the bound holds for some general class of theories beyond this, and if so for which class of theories. Note that apart from the field-

theoretic calculations based on AdS/CFT, there is no reliable method to calculate  $\eta/s$  for *any* strongly coupled quantum fluid, yet it is this class of fluids for which one expects the smallest values of  $\eta/s$ . The optimistic view is that there could exist a very general property of some large class of quantum fluids, namely, the  $\eta/s$  bound, which was unnoticed prior to the AdS/CFT calculations in large measure because there was no tractable way to compute the entropy and viscosity properties for strongly coupled theories. Of course, nature itself is an excellent analog computer, and one way to probe whether there is a bound which applies to the class of theories that describe the real world is to ask whether there are any known fluids which violate the putative bound. In ref. [2], KSS examined a number of real life fluids, including liquid helium, liquid nitrogen, and water, under a variety of conditions and found no examples where the bound was violated. Typically, the ratio  $\eta/s$  for these fluids was found to be orders of magnitude larger than the bound. This empirical data appears to be one of the strongest pieces of evidence for the existence of a bound.

Additionally, a more heuristic argument can be made for the existence of a bound [2, 23]. Consider a relatively dilute fluid which for simplicity is composed of one type of particle. By dilute we mean: i) that the dynamics of the system is dominated by two-body scattering, and ii) that the mean-free path  $l$  between collisions is much larger than both the thermal wavelength  $\lambda_T$  of the system and the characteristic range of the interaction. In effect, this dilute regime is weakly coupled from the point of view of many-body physics; quantum many-body effects are unimportant. This is the regime which can be accurately described via a Boltzmann equation [24]. A simple kinetic theory estimate of the shear viscosity in this regime was derived long ago by Maxwell:

$$\eta \sim n p_T l \sim \frac{p_T}{\sigma} \quad (2.1)$$

where  $n$  is the density and  $p_T = 2\pi/\lambda_T$  is the thermal momentum, and we have used the dilute-gas relation  $n\sigma l \sim 1$ , where  $\sigma$  is the scattering cross-section at thermal energies [24]. (For a nonrelativistic system  $p_T \sim (mT)^{1/2}$ , while for a relativistic system  $p_T \sim T$ .) In the dilute regime, the entropy density is well approximated by the free gas entropy density, and up to logarithmic corrections in  $m$  and  $T$  the entropy density  $s$  is just proportion to the density. Combining these relations allows us to write the ration  $\eta/s$  as

$$\frac{\eta}{s} \sim \frac{p_T}{n\sigma}. \quad (2.2)$$

Clearly, the expression for  $\eta/s$  in eq. (2.2) is monotonically decreasing with  $n$ . At first glance one might think that by simply increasing  $n$  one can reduce  $\eta/s$  to as small a value as one likes. However, eq. (2.2) is only a useful estimate in the dilute limit. Increasing the density, the mean free path shrinks, and eventually becomes comparable to either the range of the interaction or the thermal wavelength. Beyond this point, quantum effects alter the analysis, and one enters a strongly coupled regime. Presumably, these quantum many-body effects cause the ratio of  $\eta/s$  to stop decreasing and begin increasing. From these simple scaling arguments, it is easy to see that the density for  $l \sim \lambda_T$  occurs when  $\eta/s \sim 1$ . Therefore, at the length scale for which the quantum many-body effects are expected to begin to increase the ratio  $\eta/s$ , the effective minimum (and hence the lower bound) of  $\eta/s$  is on the order of 1.

<p>I. Class of Underlying Theories</p> <ol style="list-style-type: none"> <li>1. Any quantum mechanical system.</li> <li>2. Any nonrelativistic quantum mechanical system with one component of spin 0 or 1/2.</li> <li>3. Any “sensible” quantum field theory. <ol style="list-style-type: none"> <li>3'. Any “sensible” quantum field theory with <math>\mu = 0</math>.</li> </ol> </li> </ol> <p>II. Stability Class of Fluids</p> <ol style="list-style-type: none"> <li>a. Absolutely stable fluids only</li> <li>b. Metastable and stable fluids</li> </ol>
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**Table 1:** Classification of the many forms of the conjectured bound for the ratio of shear viscosity to entropy density.

Such general scaling and uncertainty arguments suggest that for any given system the minimum value of  $\eta/s$  will likely be of order unity (or larger if the thermal wavelength is shorter than the range of the interaction). This argument is heuristic and does not explain why the number of order unity should be  $(4\pi)^{-1}$ , but it is certainly consistent with it. A somewhat more sophisticated version of this argument may be found in ref. [23].

### 3. Classification of the variants of the KSS conjecture

To discuss the various versions of the KSS conjecture systematically, it is useful to classify the possible domains of validity of the bound. In doing so we focus on two distinguishable aspects of the domains of validity. The first aspect is the type of theory for which the conjecture is supposed to apply. The bound was originally found in a very limited class of theories — large  $N_c$  gauge theories with super-gravity duals — and assumed to hold for a broader class of theories. Thus, the first matter that we need to characterize are the classes of theories for which the bound may hold. The second aspect to be characterized is the degree of stability of a fluid described by some given class of underlying dynamical theory. In particular, this second classification delineates whether the bound is to be taken to hold for stable fluids only, or for long-lived metastable fluids as well.

Table 1 outlines a set of possible categories for both of the above aspects of the domain of validity for the bound. The listing of theory classes is intended to be ordered, more or less, in decreasing scope: i.e., as one descends the list, the possible number of fluids which can be described by each subsequent set of theories decreases.

Using the classifications delineated in table 1, each variant of the conjecture can be labelled by a pair of characters, one chosen from the list of classes of underlying theories, and another chosen from the list of stability classes. For example, if one takes the original form of the conjecture that the bound applies to “all fluids” to mean that it applies to all fluids described by quantum mechanics, then the conjecture is of class 1a or 1b, depending on whether one wishes to restrict the domain of validity to absolutely stable fluids or not. Note that the list of theory classes described in table 1 may not be an exhaustive one, but it is intended to include the natural interpretations of previously published variants of the conjecture and some modest extensions thereof.

In the next two subsections, we will further examine the classes of theories and the fluid stability classes to which the bound might apply. The first subsection will discuss the applicability of the conjecture to the various classes of the theories that we have delineated in table 1. The second subsection will explore the issue of stable versus metastable fluids. After that, we will discuss the applicability of the various versions of the conjecture to different realistic fluids.

### 3.1 Classes of theories

A fluid can be described theoretically as a many-body system whose constituent particles are mobile enough to sample the complete position space of the fluid. We can define a “theoretical fluid” by defining the interactions between particles that make up the fluid. Of course, real fluids may be regarded as theoretical fluids as well — they are the theoretical fluids associated with the correct theory of nature. The logic of the KSS conjecture is that the  $\eta/s$  bound, which was discovered in the context of gauge theories with super-gravity duals, applies to a broader class of theories. Part I of table 1 lists a number of possible classes of theories for which the KSS bound might be taken to apply.

The list of classes of theories may seem somewhat peculiar. It was generated in part to reflect the possible ways to interpret the variants of the KSS conjecture on the market. There is another reason to consider these classes. In many ways, the most natural class of theory to consider is class 1, the general class of systems describable by quantum mechanics. All of the heuristic arguments in section 2 in support of a generalization of the KSS conjecture to theories beyond those described by AdS/CFT at large  $N_c$  apply if the generalization is to generic quantum mechanical systems. (As we will see later, this is not true of any of the alternatives.) However, it is easy to see (by explicitly constructing counter-examples) that this variant in its full generality *cannot* be correct. The other classes of theories in table 1 may be thought of as ways to restrict these classes of theories to which the bound should apply in order to evade the problems with class 1.

As was briefly noted early on by KSS [2] and subsequently addressed in more detail in refs. [6] and [13], the bound may be violated by considering a nonrelativistic fluid composed of an extremely large number of distinct species, which are all degenerate in mass, and interact with each other via identical interactions. The key point is that by increasing the number of species while keeping the total density of particles and temperature fixed, the shear viscosity  $\eta$  is left essentially the same as in a single species fluid, while the entropy grows through the Gibbs mixing entropy. By making the number of species exponentially large, the bound can be violated. A detailed discussion of how this works is given in section 4.

The variants of the conjecture considered by KSS in ref. [2] are essentially those in classes 2 and 3'. These evade the problem of Gibbs mixing entropy in very different ways. Class 2 does this by explicitly limiting the number of species in the fluid to no more than 2 (the number of spin states of a spin-1/2 system), and thereby appears to restrict the growth in Gibbs entropy. Class 3' does this indirectly by restricting the chemical potentials to zero: with a zero chemical potential and a quantum field theoretic system, one cannot independently adjust the density of each of the particle species, since each particle density

is fixed by the temperature and masses of the particle. Thus by adding species at a fixed temperature, one necessarily changes the total density of particles.

There is a subtlety associated with the theories of class 2. The issue concerns the precise definition of a fluid with “one component.” Suppose that we have a many-body system with one type of particle,  $A$ , which interacts through some two-body potential. Suppose further that this interaction is attractive and some number of two-body (and/or many-body) bound states (molecules) exist. One might wish to regard a fluid composed of particles of type  $A$  as a fluid with one component, since ultimately everything in the fluid is composed of one type of particle. However, the kinetic degrees of freedom whose motion describes the fluid include both the atoms and the molecules, and the system is effectively a multi-component fluid. Furthermore, one may naively suggest that fluids which are composed from only one type of molecule, such as water, may be considered as having a single species. However, water molecules (and many other molecules) have rotational and vibrational excitations which can be accessible. These excitations cause the fluid to act like a multi-species fluid with each excited state behaving as a distinct species. For the purposes of the discussion here, we will therefore consider a system to be of “one component” only if either of the following two conditions are met: first, the particles making up the fluid do not form bound states, and second, the internal excitation energies of the particles are sufficiently high so that the excited states are not populated due to the temperature. Otherwise, we will consider the system to be a multi-component fluid.

Variants associated with class 3, which limit the conjectured bound’s applicability to systems described by “sensible quantum field theories,” attempt to evade the entropy problem in a more subtle way. The modifier “sensible” was introduced in this context by Son and Starinets [3]. In this context, the term “sensible” might be taken as a synonym for “well defined” — that is, a quantum field theory in which, at least in principle, all observables may be calculated without additional *ad hoc* input. From a practical perspective, the term “sensible” may be taken to refer to UV-complete theories; namely, those which are sensible down to arbitrarily short distances, and thus do not require additional prescriptions for dealing with uncontrolled short-distance physics. Thus, “sensible quantum field theories” would be taken to include asymptotically free field theories such as QCD or conformal field theories. It should be noted here that the classes of theories believed to be UV-complete are rather limited. Many renormalizable theories with which we have considerable experience are probably not “sensible” (at least perturbatively) in the sense used here. For example, theories such as QED and linear sigma models are presumably not “sensible” in that it is generally thought that unless they are trivial, they are likely to be ill defined in the ultraviolet.

How might the restriction to “sensible” quantum field theories possibly evade the difficulty posed by Gibbs mixing entropy? Recall that the violation of the bound may well require an extremely large number of essentially identical species of nonrelativistic particles. Accordingly, it is difficult to find any realistic situation where it occurs for real world



fluids.<sup>1</sup> One might hope that the difficulty of constructing practical examples of such fluids might actually reflect some deep and previously undiscovered principle. This hypothetical principle must go beyond that which is contained implicitly in quantum mechanics, since quantum mechanical systems can be found which violate the bound in eq. (1.1). Thus, it is natural to ask if such a principle could have a quantum field theoretic origin. This gains some credence from the fact that the conjectured bound was first seen in a particular class of “sensible” quantum field theories (conformal field theories with gravity duals). Thus, one might speculate that the bound should only apply to systems which are ultimately described by sensible quantum field theories, and therefore it should not be possible to find a UV-complete field theory that can give rise to a system that can violate the KSS bound.

On its face, it seems quite implausible that constraining the relativistic field theory underlying a non-relativistic fluid to be UV-complete should somehow rule out nonrelativistic fluids of many components which violate the bound in eq. (1.1) through a very large Gibbs mixing entropy. After all, the short distance dynamics of the underlying quantum field theory typically occur on radically different scales than the scales of the effective degrees of freedom in the nonrelativistic gases of interest. Accordingly, it is very difficult to see how a constraint on the dynamics on  $\eta/s$  for the fluid can arise naturally. Moreover, as noted in ref. [1], even after units are restored the speed of light does not appear in the bound. Thus, it is very hard to understand how the origin of the bound could be related to the relativistic nature of the underlying field theory.

The preceding arguments suggest that it is very hard understand from first principles why a restriction to “sensible” relativistic field theories ought to yield the bound. However, naive attempts to increase the number of nonrelativistic species of particles in a gas by increasing the number of types of particles in the underlying quantum field theory can easily cause a theory to lose asymptotic freedom and thereby ceasing to be a “sensible” quantum field theory [6, 7].

It is useful to illustrate how this can happen. Let us consider a nonrelativistic gas which is predominantly composed of one type of pion of mass  $m_\pi$ , for instance the  $\pi^+$ . Such a gas undoubtedly has its origins in QCD, a UV-complete quantum field theory. To describe such a gas in the context of QCD, we can consider the theory at a finite temperature  $T$ , and a chemical potential  $\mu_u$  for the up quark  $u$  of the form  $\mu_u \bar{u} \gamma_0 u$ . (It is unnecessary to also impose a chemical potential for the down quarks.) If the system is in the regime  $T \ll m_\pi$  and  $\Lambda \gg \mu_u > m_\pi$  where  $\Lambda$  is a typical hadronic scale of order 1 GeV, then it is essentially a nonrelativistic gas of  $\pi^+$  mesons. Now suppose that we wish to generalize this to a many-species pion gas. To do this, let us generalize QCD to include  $N_f$  degenerate flavors of quarks with  $N_f$  large and even. Suppose we add a common chemical potential  $\mu_c$  for half of the flavors:

$$\sum_{j=1}^{N_f/2} \mu_c \bar{q}_j \gamma_0 q_j \tag{3.1}$$

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<sup>1</sup>Note, however, that ref. [13] explores the possibility that it may be possible to violate the KSS bound with a gas of fullerenes

while keeping  $T \ll m_\pi$ . This will create a nonrelativistic system containing  $N_f^2/4$  types of pions (each one with a quark of type  $q_j$  with  $j \leq N_f/2$  and an anti-quark of type  $\bar{q}_k$  with  $k > N_f/2$ ). By carefully tuning  $\mu_c$  while increasing  $N_f$ , the total density of pions can be kept fixed while increasing the number of species. This appears to allow one to create the conditions in which the Gibbs entropy dominates the ratio of  $\eta/s$  and causes a violation of the KSS bound.

However, there is a catch. Recall that for small  $g$ , the beta function for QCD is given by

$$\beta(g) = -\frac{g^3}{16\pi^2} \left( \frac{11N_c}{3} - \frac{2N_f}{3} \right). \quad (3.2)$$

Asymptotic freedom requires that  $11N_c > 2N_f$ . By increasing  $N_f$  in order to violate the bound in eq. (1.1), the underlying theory is pushed outside of the domain of “sensible” theories. Of course, one might try to evade this by increasing  $N_c$  at the same time as one increases  $N_f$ ; by fixing the ratio  $N_c/N_f$  as the large  $N_f$  limit is taken, asymptotic freedom can be maintained. However, recall that the cross section for  $\pi - \pi$  scattering scales as  $1/N_c^2 \sim 1/N_f^2$  [26]. For a weakly interacting fluid, the shear viscosity is expected to scale with the inverse of the cross-section [24]. Thus, by increasing  $N_c$  along with  $N_f$  to maintain asymptotic freedom and keep the theory sensible, one finds that  $\eta \sim N_f^2$ . On the other hand, the Gibbs mixing entropy grows only with  $\log(N_f)$ , so  $\eta/s \sim N_f^2/\log(N_f)$  for large  $N_f$ . As a result, in a pion gas in the large number of species limit, the decrease in the cross section associated with the  $N_c$  scaling necessary to maintain asymptotic freedom overwhelms the increase in Gibbs mixing entropy due to the  $N_f$  scaling, and  $\eta/s$  is driven to infinity in the combined  $N_f \sim N_c \rightarrow \infty$  limit.

The example of pion gases in QCD shows how the restriction to a “sensible” theory can prevent the system from ever getting into a regime where the Gibbs mixing entropy dominates the ratio of  $\eta/s$  and thus violates the KSS bound. The central question underlying the theories associated with class 3 is whether the situation seen for pion gases in QCD is paradigmatic for all sensible theories.

There is an additional important subtlety associated with the notion of “sensible” in class 3; namely, whether the standard model should be regarded as a sensible quantum field theory. The standard model contains scalar fields and is probably *not* UV-complete. This implies that class 3 should not apply to the standard model *per se*. However, the standard model may be regarded as the low energy effective theory for some theory (a field theory, a string theory, or something else) which *must* make sense in the ultraviolet since it describes nature. Thus, it might be useful to regard “the standard model,” as described in the textbooks, to include the appropriate UV-completion for real world situations, and hence be “sensible.”

Having delineated some of the possible domains of validity of the conjectured bound on  $\eta/s$ , in the next subsection we will discuss the matter to which stability classes of fluids the conjectured bound may apply.

### 3.2 Metastability

In addition to distinguishing variants of the conjecture according to the classes of underlying

theories to which they apply, we also need to discuss the stability classes of the fluids for which the  $\eta/s$  bound may apply. A fluid can be described as either stable or metastable. In this subsection, we will examine some of the issues associated with the applicability of the bound to stable and metastable fluids.

We are defining a metastable fluid to be one which is in a macroscopic state which is not the state of lowest free energy; such a fluid is expected to decay over time to the true macroscopic ground state. If the time scale of the decay is extremely long compared to other relevant time scales, the fluid is considered to be metastable. A stable fluid, on the other hand, is one in which no decay is possible, i.e., the fluid is in its ground state and will remain there unless it is perturbed. Metastable fluids are characterized by at least two relevant time scales. First, there is  $\tau_{\text{fl}}$ , which is the longest microscopic time scale relevant for fluid motion. In practice, for a typical real world fluid,  $\tau_{\text{fl}}$  might be taken to be several times the characteristic collision time between molecules. Thus,  $\tau_{\text{fl}}$  characterizes the minimum time scale for which it is meaningful to talk about macroscopic fluid behavior. Next, there is the time scale  $\tau_{\text{meta}}$  for the decay from a metastable fluid to a stable (that is, lowest-energy) configuration.

The characterization of the fluid clearly depends on the ratio  $\tau_{\text{fl}}/\tau_{\text{meta}}$ . If  $\tau_{\text{meta}}/\tau_{\text{fl}} \sim 1$  or less, then the decay time is of the same order or less than the characteristic time for fluid-like behavior, and it is not meaningful to describe the system as being in a well-characterized fluid state. In effect, in such a regime the fluid is so unstable that one cannot measure properties like shear viscosity or entropy before the system decays into a qualitatively different type of fluid. However, if  $\tau_{\text{meta}}/\tau_{\text{fl}} \gg 1$ , the decay time scale is much longer than the time scale of the measurements needed to determine fluid properties such as the shear viscosity. In this case, the fluid can be said to be metastable, and properties such as viscosity and entropy are essentially well defined in the metastable phase. For an extremely large  $\tau_{\text{meta}}$ , the metastable fluid acts to a very good approximation as if it were a stable fluid. We should note that many systems which we obviously characterize as fluids in the real world are actually metastable. An extreme example is nitroglycerin ( $C_3H_5(NO_3)_3$ ). Above its melting point of  $13.2^\circ\text{C}$  it is clearly a fluid — it will slosh around in a beaker. However, liquid nitroglycerin is obviously *not* in a configuration at the minimum of the free energy — considerable energy can be released when the molecules break up and rearrange. It is noteworthy that in the real world,  $\tau_{\text{meta}}$  for metastable fluids is typically *many* orders of magnitude larger than  $\tau_{\text{fl}}$ .

There are two ways in which a system can be metastable in the sense used here. The first is the rather typical example in statistical physics in which a macroscopic phase is locally stable while being globally unstable. That is, any small fluctuation of a macroscopic fluid property (e.g., density) from its value in the metastable phase increases the free energy, but large fluctuations can lower it. This is quite familiar in systems which can undergo first-order phase transitions. The system can be beyond the phase transition point but stay in the old phase. Thus, for example, water may be supercooled or the relative humidity can be greater than 100%. Such systems can live for a very long time (if undisturbed) since there is barrier which must be either surmounted via thermal fluctuation or tunneled through quantum mechanically. In either case, if the barriers are large, the lifetimes of the

metastable phases grow exponentially.

There is a second way for a system to be metastable. A system can be locally unstable in terms of *some* thermodynamic variables, but the time scale associated with the local instability can be very long. It is this sort of metastability which is relevant for many of the discussions in this paper. For example, this can happen in chemical systems. A system can be in thermal equilibrium kinetically but not chemically; however, the time scale for reaching chemical equilibrium can be very large. Suppose, for example, that one initially has a gas composed of molecules of one type,  $A$ . Suppose further that the reaction  $A + A \rightarrow B + C$  (where  $B$  and  $C$  are two other types of molecules) is exothermic, but the reaction rate is *very* small compared to the rate of elastic scattering of particles of type  $A$ . This will happen if the activation energy for the reaction is well above the temperature. In such a case, over very long time scales the system will act like a fluid of molecules of type  $A$  in thermal equilibrium kinetically, despite being out of thermal equilibrium chemically. Locally, as well as globally, the system is not at a minimum of the free energy for all of the thermodynamic degrees of freedom, but nonetheless behaves like a fluid.

We noted above that when  $\tau_{\text{meta}} \gg \tau_{\text{fl}}$ , fluid properties such as shear viscosity are essentially well defined. In a strict sense, however, they are not. As a matter of principle, transport properties, such as shear viscosity, describe the linear response of a fluid to a perturbation. This response is dynamical, and takes a certain characteristic time to play out. We can identify this time as  $\tau_{\text{fl}}$ . The transport properties are only well defined to the extent that the underlying fluid does not change its nature over this dynamical time scale. Since a metastable fluid *does* change its properties over time, there is an intrinsic ambiguity in any evaluation of  $\eta$ . One might expect that any uncertainty in the value of  $\eta$  is roughly of relative order  $\tau_{\text{fl}}/\tau_{\text{meta}}$ . Fortunately, in a good metastable system this is an exceptionally small number, and the ambiguity is very small.

The issue of metastable fluids is important in the context of the KSS conjecture. The central question is whether the conjectured bound applies to metastable fluids as well as to stable fluids. This may seem like a relatively minor issue if the bound applies to the theories in class 1. Then the question of whether the bound applies to metastable fluids reduces to the issue of whether it applies to normal stable fluids such as water, or whether it also applies to metastable fluids such as nitroglycerine. However, as will become apparent in the next subsection, if the bound only applies to theories in class 3, the question of whether the bound applies to metastable fluids determines the bound applicability to familiar real-world fluids.

To the extent that the KSS conjecture somehow captures an essential property that a system needs to possess to behave as a fluid, one might naturally assume that it should also apply to metastable systems whose macroscopic behavior is clearly that of a fluid. There is an objection of principle that could be made here, in that the conjecture is sharp — it provides a definite bound for  $\eta/s$  — while the quantities  $\eta$  and  $s$  are intrinsically ambiguous for a metastable fluid. Of course, as noted above the ambiguities are very small for long-lived metastable fluids. Accordingly, it is highly plausible that the conjecture, if correct, applies to metastable fluids with one minor alteration: the bound may be slightly violated, but all possible violations must be within the scales of ambiguities of the quantities. In

Variant	QGP	$He$	$H_2O$	$C_3H_5(NO_3)_3$
1a.	Y	Y	Y	N
1b.	Y	Y	Y	Y
2a.	N	Y	N	N
2b.	N	Y	N	N
3a.	Y	N	N	N
3b.	Y	Y	Y	Y
3'.	Y	N	N	N

**Table 2:** Table showing if each variant of the conjecture can be applied (at least approximately) to either the quark gluon plasma (QGP), liquid helium ( $He$ ), water ( $H_2O$ ), liquid nitroglycerin ( $C_3H_5(NO_3)_3$ ); Y(es), N(o)

practice, for real metastable fluids, these violations are extraordinarily small, and as a practical matter the bound would then be taken to hold for any long-lived metastable fluid. We generally take the view that is unnatural for there to be a fundamental property which applies to all stable fluids, but which does not apply — even approximately — to metastable fluids no matter how long-lived. It seems far more natural to assume that in the limit of infinite lifetime, a metastable fluid would be indistinguishable from a stable one, and that it would share all of the essential properties of stable fluids. Having said this, as a logical matter it is certainly possible that the bound only applies — even approximately — only to absolutely stable fluids. Accordingly, it is important to classify the possible  $\eta/s$  conjectures according to whether or not they apply to metastable systems.

### 3.3 Applicability of the various classes to real fluids

Having enumerated various forms of the conjecture, it is important to see the types of realistic fluids to which they apply. In table 2, we show the applicability of the various forms of the conjecture to four different types of fluids which serve to illustrate the broad issues of where the various classes apply. The fluids we examine — the quark-gluon plasma, liquid helium, water, and nitroglycerine — were chosen to serve as paradigms for broad classes of fluids.

First, consider the quark-gluon plasma. It is generally believed that the dynamics of high energy heavy ion collisions depend essentially on QCD alone — *i.e.*, electroweak effects are small. Moreover, it is generally thought that the system thermalizes, at least approximately, over reasonably large spatial regions, and that in these regions the net baryon density is low since the bulk of the baryon number goes down the beam pipe. Thus, to a good approximation these regions are well described by QCD at finite temperature and zero chemical potential. If the temperature in these regions is large enough (above  $\sim 170$  MeV), these regions can be said to contain a quark-gluon plasma. Note that in saying this we do not necessarily imply that QCD has undergone a phase transition into a quark-gluon plasma phase; a rapid cross over into a qualitatively high-temperature regime is adequate.

Clearly, nontrivial approximations are needed in order to describe this physical system in terms of thermalized QCD at zero chemical potential. However, if one accepts these approximations as valid — as we will do implicitly for the purpose of this discussion — then one has a well-characterized field theoretic description of the quark-gluon plasma. Within that characterization, it is clear that all of the variants of the conjecture should apply to this system, except for the variant with theories of class 2. As a quantum field theory, it is certainly a quantum mechanical system, and thus falls neatly into class 1. QCD is the archetypical example of a “sensible” field theory: it is asymptotically free, and hence is described by theories of class 3. Moreover, as a system at zero chemical potential, it falls into class 3′. Clearly, since the quark-gluon plasma is a relativistic system with many components, it does not fit into class 2.

Next, consider liquid water, which is truly an archetypical example of a fluid. Clearly from the perspective of chemical interactions, water is a stable fluid. One could model water to very high accuracy using a many-body quantum-mechanical description based on electrons, oxygen nuclei and hydrogen nuclei as the basic degrees of freedom, interacting via a Coulomb potential and (small) magnetic moment interactions. While in practice, it would be very hard to compute  $\eta/s$  from such a model, in principle it *is* computable, and we have every reason to believe that such a description would be very accurate. Thus, one expects that variants 1a and 1b of the conjecture should apply to water.

However, one does not expect variants of class 2 to apply. The previous description based on electrons and nuclei clearly violates the condition that there is only one component to the fluid. One might try to avoid this by considering an effective quantum mechanical model of the dynamics of water where the fundamental building blocks are water molecules interacting via effective interactions. Such a description would be under the umbrella of class 2 provided that only a single internal quantum state of the water molecule was relevant to the dynamics. However, the minimum excitation energy of a water molecule is 0.16 K [27], which corresponds to a rotational level, while for liquid water  $T > 273\text{K}$ . Thus, in practice, water molecules are not to be found predominantly in their lowest energy level in liquid water — many rotation levels of the molecules are excited, and the system does not act like a single-component fluid.

The applicability of conjectures based on theories of class 3 (“sensible” quantum field theories) to water is subtle and perhaps somewhat counterintuitive. Since water is a real world fluid and thus is presumably described by the standard model — a quantum field theory — it seems natural that water be included within the variants of the conjecture based on class 3. As noted above, there is some question as to whether we should consider the standard model to be a “sensible” quantum field theory, but for the moment let us assume that it is legitimate to do so. With this assumption, it may seem obvious that variant 3a applies to water since water is a stable fluid. However, this is not the case. Although it is stable chemically, water is *not* stable under the dynamics of the standard model: nuclear reactions are part of the standard model and can alter the constituents of water. For example, it is energetically allowable for two of the hydrogen nuclei in water to fuse in the reaction  $p+p \rightarrow d+e^++\nu_e$ . Of course, the decay time of nuclear fusion in water is *very* long, indeed much longer than a Hubble time. The reason for this is simply that

the Coulomb barrier is very large compared to thermal energies, and the rate of thermal fusion is thus exponentially small. Thus from the perspective of the standard model water is metastable rather than stable: variant 3a of the conjecture does not apply to water, but variant 3b does, at least to the extent that we can consider the standard model, and whatever lies beyond it, as a sensible quantum field theory. Clearly, theories of class 3' do not apply to water since this class is a subclass of 3a.

The fact that water is not stable under the dynamics of the standard model reflects the conservation laws of the standard model. Clearly, under the standard model the number of hydrogen and oxygen nuclei do not represent conserved quantities. Apart from electric charge, the only global conserved quantity in the standard model is  $B - L$ ; due to anomalies the baryon number  $B$  and lepton number  $L$  are not separately conserved. Thus, the only type of stable fluid we can specify in the standard model is one with a fixed chemical potential for  $B - L$ .

One might argue that the rates of nuclear reactions are so slow that they could not possibly be relevant to the validity of the conjecture. While this is a very plausible argument, it is simply an argument against a requirement that the conjectured bound needs absolute stability rather than metastability.

There is an alternative argument which can be made that variant 3a can apply to water [25]. Nuclear reactions are totally irrelevant at the scale of interest for water. Thus, to study water one might replace the standard model with a variant of quantum electrodynamics containing electrons and fundamental fields representing the proton and the oxygen nucleus. To the extent that hyperfine effects involving the nuclear spins are unimportant to the dynamics of water, such a system will behave like water and will be absolutely stable, apparently putting water in the domain of variant 3a. However, there is a problem with this setup: QED is not asymptotically free and as a result it is presumably not “sensible”. One might hope to evade this by embedding this low energy QED-like theory into another theory which a) *is* asymptotically free, b) leaves the low-energy QED physics essentially unaltered, and c) does not introduce any instabilities for water. Unfortunately, it is by no means clear that it is possible to find any field theories which meet these criteria. Until such a theory is constructed, we will take the view that variant 3a should not be regarded as applying to water.

Other real world fluids dominated by chemical (*i.e.*, electromagnetic) interactions are similar to water in terms of their classification, with obvious modifications. Thus, for example, liquid helium is like water in being described by variants 1a, 1b, 3b, and not 3a. It differs in that the lowest excitations for helium are electronic in nature, since helium (unlike water) is an atomic fluid as opposed to a molecular fluid. Since liquid helium temperatures are well below the excitation energy for electronic transitions, the atoms in liquid helium are essentially all found in their ground state. Thus, it is possible to model liquid helium with good accuracy in terms of a quantum mechanical many-body system with fundamental helium atom degrees of freedom interacting via an effective potential. Within the framework of such a model, liquid helium, unlike water, falls within the domain conjectures of classes 2a and 2b. Similarly, nitroglycerine is like water in terms of the variants of the conjecture which describe it, with the exception of class 1a which describes water (which is a stable

fluid chemically) but not nitroglycerine (which is obviously metastable).

Having discussed a framework for labeling the possible variants of the  $\eta/s$  bound conjecture, in the next three sections we will construct and discuss counterexamples to variants of the conjecture of classes 1, 2, and 3. We will ultimately show that that only class 3a (and its subclass 3') remains viable.

#### 4. Class 1

The first variant of the KSS conjecture that we will closely examine is class 1: the conjecture that  $\eta/s \geq 1/4\pi$  for all fluids described by quantum mechanics. This variant seems to be very close to the original form of the bound proposed by KSS [1]. Note that this variant of the conjecture has much stronger support than the other variants; all of the heuristic arguments as well as all of the empirical evidence given in support of the KSS bound support this variant. This variant has the widest applicability, as it applies to any fluid, both relativistic and nonrelativistic ones, and both physically realizable or purely theoretical fluids provided they are described by quantum mechanics.

However, ref. [2] and others [6, 13] have noted that this variant of the conjectured bound can be violated by considering a fluid with a large number of different species. In this section, we elaborate on the previous arguments of ref. [6] to describe a nonrelativistic quantum mechanical system which violates the conjectured bound.

##### 4.1 A nonrelativistic gas

Reference [6] considers a nonrelativistic quantum many-body system with a large number of species for which the computation of the ratio  $\eta/s$  is analytically tractable, up to corrections which can be made arbitrarily small. By imposing a particular set of scaling relations on the parameters of the system, it is possible to demonstrate that  $\eta/s$  can violate variants 1a and 1b in the limit of a large number of species. We review this argument here.

Consider a gas composed of a number ( $N_s$ ) of distinct species of spin-0 bosons of degenerate mass,  $m$ , which can interact via a two-body potential. The two-body potential is identical for all species, but is limited to a finite range,  $R$ . The gas is in thermal equilibrium at a temperature  $T$ , and has the same density for each species,  $n_a = n/N_s$ , where  $n$  is the overall density of the system. The system is in a low temperature and low density regime such that

$$R^{-2}, a^{-2} \gg mT \gg n^{2/3}, \tag{4.1}$$

where  $a$  is the scattering length, and  $mT$  is the thermal momentum squared. This regime can be maintained by using the following scaling of the density and temperature:

$$n = \frac{n_0}{\xi^4} \quad T = \frac{T_0}{\xi^2}, \tag{4.2}$$

where  $n_0$  and  $T_0$  are independent of the dimensionless scaling parameter  $\xi$ . With a sufficiently large value for  $\xi$ , eq. (4.1) can be easily satisfied.

In this density and temperature regime, the entropy for the system is simply that of a classical ideal gas, with small corrections. The key point is that the temperature is high



enough relative to  $n_0^{2/3}/m$  for the classical expression to hold, while the density is low enough to neglect the interactions. The entropy density can then be written in terms of the scaling in eq. (4.2) as

$$s \simeq n_0 \left( \log \left( \frac{(mT_0)^{3/2}}{n_0} \right) + \frac{5}{2} + \log(\xi) + \log(N_s) \right), \quad (4.3)$$

where the term  $\log(N_s)$  is associated with the Gibbs mixing entropy of the  $N_s$  different species.

Furthermore, in this density and temperature regime, the thermal wavelength is much shorter than the inter-particle spacing, meaning that the many-body dynamics are essentially classical. Moreover, the low density implies that the many-body dynamics are dominated by binary collisions, implying that the system is in the regime of validity for the Boltzmann equation [24]. The low temperature further implies that the two-body collisions are dominated by s-wave scattering, with a cross section essentially unchanged from its zero momentum value. That is, two-body scattering in this system can be approximated as isotropic and energy independent, which is formally the same as classical hard sphere scattering.

The shear viscosity is analytically calculable in such a system [24], and it is given by  $\eta = C_{\text{hs}} \sqrt{mT}/d^2$ , where  $d$  is the diameter of the hard spheres, and  $C_{\text{hs}} \approx .179$  is a coefficient that is numerically calculable.<sup>2</sup> Identifying the scattering length  $a$  as the effective hard sphere diameter, we can now calculate the ratio  $\eta/s$ :

$$\frac{\eta}{s} \simeq \frac{C_{\text{hs}} \xi^3 \sqrt{mT_0}}{a^2 n_0 \left( \log \left( \frac{(mT_0)^{3/2}}{n_0} \right) + \frac{5}{2} + \log(\xi) + \log(N_s) \right)}. \quad (4.4)$$

Corrections to eq. (4.4) are suppressed by powers of  $1/\xi$  and should become irrelevant for sufficiently large  $\xi$ .

The derivation of eq. (4.4) required the system to be in a low density and low temperature regime such that a classical approximation for both the shear viscosity and the entropy density can be made. This limit does not place any constraints on the number of species of particles in the fluid. Accordingly, one can demand that the number of species scale exponentially with the scaling parameter:

$$N_s = \exp(\xi^4) \quad (4.5)$$

As the temperature and density decrease, the number of species increases simultaneously. When eqs. (4.4) and (4.5) are combined, the large  $\xi$  scaling of the ratio is

$$\frac{\eta}{s} \simeq \frac{1}{\xi} \frac{C_{\text{hs}} \sqrt{mT_0}}{a^2 n_0} \quad (4.6)$$

up to power law corrections in  $1/\xi$ . Clearly, in this combined limit, the ratio  $\eta/s$  can violate the conjectured bound simply by making  $\xi$  sufficiently large. This violation stems completely from the large Gibbs mixing entropy associated with the exponentially large number of species.

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<sup>2</sup>The coefficient,  $C_{\text{hs}}$ , can be calculated in the Chapman Enskog expansion as outlined in [24] and is  $5/(16\sqrt{\pi})$  to leading order.

## 4.2 Stability

In this subsection, we will discuss the stability class of the fluid that we have described above. The argument in the preceding section does not depend on the interparticle potential and thus will continue to hold for any choice of the interparticle potential. If we choose the interparticle potential to be purely repulsive, the particles making up the fluid cannot lower their energies by forming bound states. Therefore, with this choice, the system that we have described above is a stable fluid with an arbitrarily small value of the ratio  $\eta/s$ . This is sufficient to demonstrate that this system is a counterexample to *both* class 1a and 1b variants of the conjecture.

While with the system above we were free to choose the interaction potential to be whatever we wanted, in some other situations this is not possible. In particular, in our discussion of systems of class 3, in section 6 we will find that the interaction potential there will necessarily be an attractive one. To see the implications on the stability of a fluid of an interaction potential with some attractive regions in a simple context, we will now discuss the consequences of choosing an interparticle potential with some attractive regions for the system in the previous section.

One might worry that with such a potential, the fluid could lower its energy by forming bound states, or by “clumping” together; that is, by forming macroscopic regions of higher density where the attraction is enhanced and the free energy is lowered. If either situation is possible, the fluid would then be either unstable or metastable. As discussed in section 3.2, in order to distinguish between these two cases, we need to compare  $\tau_{\text{met}}$ , the characteristic time for the phase to change macroscopically, with  $\tau_{\text{fl}}$ . We can show that in our scaling regime  $\tau_{\text{meta}}/\tau_{\text{fl}}$  diverges as  $\xi^5$  or faster, ensuring that when  $\xi$  is large the system is metastable.

The type of metastability with the decay mechanism which yields the fastest possible decay parametrically is for systems which can form two-body bound states. As is well known, in a nonrelativistic gas three-body collisions are necessary to allow the formation of two-body bound states due to energy and momentum conservation. Therefore, the decay time  $\tau_{\text{met}}$  scales with the time between three-body collisions in the system. The characteristic time scale of the fluid  $\tau_{\text{fl}}$  scales with the time scale for two-body collisions. Therefore the ratio  $\tau_{\text{meta}}/\tau_{\text{fl}}$  has roughly the same scaling as  $\tau_3/\tau_2$ , where  $\tau_3$  and  $\tau_2$  are the three-body and two-body collision time scales, respectively.

The time between two-body collisions is essentially just the mean free time of particles in the fluid. The mean free time  $\tau_{\text{mf}}$  is related to the mean free path  $l_{\text{mf}}$  by

$$\tau_{\text{mf}} = l_{\text{mf}}/v, \quad (4.7)$$

where  $v$  is the rms velocity of particles in the fluid. In dilute classical gases the mean free path  $l$  can be related to the density and the interaction cross section,

$$nl_{\text{mf}}\sigma \sim 1. \quad (4.8)$$

The rms velocity  $v$  can be related to the thermal momentum associated with the fluid:  $mv \sim \sqrt{mT}$ , where  $m$  is the mass of the particle, and  $T$  is the temperature of the fluid.

Combining these equations and the scaling relations of eq. (4.2), we see that the mean free time scales like

$$\tau_{\text{mf}} = \frac{1}{n\sigma} \sqrt{\frac{m}{T}} \sim \xi^5 \frac{1}{n_0 R^2} \sqrt{\frac{m}{T_0}}, \quad (4.9)$$

where we have used the relation  $\sigma \sim R^2$ , with  $R$  being the characteristic range of the interaction.

In addition to  $\tau_{\text{mf}}$ , we must examine the characteristic time that two particles spend interacting during a collision,  $\tau_{\text{int}}$ . Equation (4.2) implies that scattering is at low momentum. As a result,  $\tau_{\text{int}}$  does not scale with  $\xi$ , since it is essentially a function of the details of the two-body potential and does not depend on  $v$ . The fraction of the time between two-body collisions during which the particles are interacting is  $f \sim \tau_{\text{int}}/\tau_2 \sim \xi^{-5}$ .

To form a two-body bound state, a three-body collision is necessary. That is, while two particles are in the process of interacting, a third particle must collide with them. In terms of the quantities defined previously, the time scale for such events is simply  $\tau_3 = \tau_2/f$ . As a result, we see that  $\tau_{\text{meta}}/\tau_{\text{fl}} \sim \tau_3/\tau_2 \sim \xi^5$ , as claimed above. Other mechanisms take longer parametrically: if the most rapid decay involves the formation of an  $N$ -body state, an analogous calculation yields  $\tau_{\text{meta}}/\tau_{\text{fl}} \sim \xi^{5(N-1)}$ .

To summarize, the arguments in this section show that the variants of the KSS bound of class 1 can be violated by a fluid with a large number of species. Depending on the choice of an interaction potential, the fluid that we have described can be either stable or metastable. While the example used to demonstrate the violation of the bound is highly artificial and unlikely to be realizable even approximately in a real world setting, as a mathematical matter it is a legitimate counterexample. The implication is that the most well-supported and most widely applicable variants of the conjecture — those of class 1 — are not tenable.

## 5. Class 2

In this section, we discuss the variants of the KSS conjecture of class 2. This form of the conjecture states that  $\eta/s \geq 1/4\pi$  for all nonrelativistic fluids composed of a single species of particle of spin-0 or spin-1/2. This variant of the conjecture is essentially the one that was proposed by KSS in ref. [2]. By restricting the number of allowable species, this variant of the conjecture attempts to avoid the problem with the Gibbs mixing entropy that allowed the construction of a counterexample to the variants of class 1.

Note at the outset that the evidence in support of this class of conjecture is quite limited. The AdS/CFT duality arguments do not apply. Since these calculations were done in the large  $N_c$  limit, it is hard to understand how they could justify a bound that fails for a large number of species and only works when the number of species is small enough. Moreover, much of the empirical evidence in favor of a KSS bound does not apply to variants of this sort. The term “single-species” in this context refers to systems whose constituents are either elementary or are in their ground state and do not access higher excited states. As a result, liquid water is not covered in this variant of the conjecture: water molecules in a liquid state can access rotational modes, making water a multi-species

fluid from this perspective. This limits the applicability of this variant of the bound mostly to mono-atomic fluids, such as liquid helium. Since the vast majority of real world fluids are not in this class, the fact that no known violation of the bound exists for real fluids provides only modest support for the bound.

In this section, we will investigate a counterexample to variants of class 2. We will give an example of a stable quantum-mechanical system composed of only one kind of spin-0 particle that can violate the KSS bound. Since the counterexample is for a stable fluid it appears to rule out both variants 2a and 2b.

To demonstrate that the existence of a class 2 system violates the bound, we first define the system by choosing a particular two-body interaction potential. The properties of the fluid in a non-relativistic regime are determined by the interaction potential along with the temperature and the density. The basic idea is to construct a two-body interaction of finite range which has an extremely large number of two-body resonant states right above threshold. We show that the entropy for such a system has a lower bound, which by a judicious choice of parameters can be made arbitrarily large, even though there is only a single species of particles making up the fluid. Finally we argue that the shear viscosity of such a system is not expected to become uncontrollably large as the parameters are adjusted to make the entropy grow arbitrarily. Thus it appears that the ratio of  $\eta/s$  can be made arbitrarily small within this class of theory.

### 5.1 Constructing the system

In this subsection, we define a single-species fluid composed of identical, stable, spin-0 particles. These identical spin-0 particles are considered to be the fundamental particles of the fluid. We will choose a finite-range two-body interaction that supports no bound states (two-body or many-body) while supporting an arbitrary number of arbitrarily low-lying resonant states in the scattering amplitude. The resonant states may be long-lived (depending on the choice of parameters of the potential), but it is important that they are indeed resonant states, and *not* bound states, so that there is no question that the fluid is of a single species.

Before discussing a detailed form of interaction which can generate this situation, it is important to note at the outset the interaction will require an exceptional degree of fine-tuning. The principal reason for this is that we require that the range of the interaction remains fixed as we add resonances. We impose this requirement because we wish to keep the density of the fluid fixed as we add resonances in order to avoid having many particles simultaneously within the range of interaction. This creates a strong constraint in which we require an exponentially large number of nearly degenerate s-wave resonances near threshold for a system of fixed spatial extent. A useful way to envision making a system of finite size with multiple nearly degenerate two-body resonances is to start by constructing a system with numerous nearly degenerate two-body s-wave bound states and then add a repulsive potential to push them into the continuum.

However, it is not trivial to create a large number of nearly degenerate bound states with the same quantum numbers due to level repulsion. One way to proceed is by using a central potential which has numerous nested spherical-shell-shaped wells; we denote the

number of wells as  $N$ . Clearly, if the spatial size of the interaction is kept fixed as one goes to a regime of large  $N$  (as is needed to achieve many bound states), the width of each well in the radial coordinate,  $r$ , must be very small. To understand the tuning of parameters that is required, it is easiest to start by considering a system with a single well at a fixed position — with the position corresponding to the positions of one of the nested wells. The parameters are picked such that the single-well system has a single two-particle bound state. This can be achieved by tuning either the width or the depth of the well, or both. Arbitrarily narrow wells can always be constructed to have a single bound state with fixed binding energy by making the well deep enough. In taking the width in the radial direction to be small (as we are forced to), in essence one is fine-tuning the depth of the potential,  $V_0$ , so that the binding energy is a very small fraction of  $V_0$ . For a generic well, it is not possible to do this for more than one bound state level. The bound state wave functions will be localized in the radial coordinate around the well. Note that there is a considerable level of parameter-tuning necessary to achieve this.

Now suppose we consider a system with all  $N$  of the wells present simultaneously. The parameters would need to be further tuned so that the bound states in each of the  $N$  wells are nearly degenerate. To the extent that bound state wave functions for the single well case were well localized — i.e., have a spreading in  $r$  which is much less than spacing between levels — the full system will have  $N$  nearly degenerate bound states, each with an energy near that of the single well case. However, if that condition is not met, there will be significant level repulsion and the condition of near degeneracy will be destroyed. The characteristic spread of the wave functions is  $(mB)^{-1/2}$  where  $m$  is the particle mass, and  $B$  is the binding energy. Accordingly, to include a large number of wells within a fixed radius while keeping the levels nearly degenerate requires that the binding energy be tuned to be large.

There is a final level of tuning required. We have shown that considerable tuning is required to get  $N$  nearly degenerate deeply bound states in a system with  $N$  nested wells with fixed range. However, we wish to have a system with  $N$  resonances. We can do this by adding a finite-range repulsive step function potential which will push the bound states just above threshold yielding resonances. As noted above, the bound states need to be very deeply bound. Accordingly, to get resonances just above threshold, one must tune the strength of the repulsive interaction to very high accuracy to cancel out the binding, leaving behind barely unbound resonances. However, in principle there is nothing to prevent one from arranging a system with all of this fine-tuning done as accurately as one wishes, yielding as many resonances as one wants as close to threshold as desired.

An example of a two-body central potential that has the desired properties is

$$V(r) = -b \sum_{k=1}^N \delta\left(r - \frac{kL}{N}\right) + V_0 \theta\left(r - \left(L + \frac{L}{N}\right)\right), \quad (5.1)$$

where  $r$  is the distance between fundamental particles,  $L$  is the range of the potential,  $b$  is the strength of each of the  $N$  delta functions, and the delta functions are raised on a potential step of height  $V_0$ . The additional factor of  $L/N$  in the step potential is intended

to extended the range of the potential just beyond the last delta function. This ensures that potential is identical in the neighborhood of each delta function. The  $\delta$  functions in the potentials should be thought of as very deep, narrow potential wells — where the details of how this is done becomes irrelevant provided the width is much smaller than all other scales in the problem. One can imagine tuning the parameters in the interaction of eq. (5.1) (that is, choosing  $b$  and  $V_0$ ) so that any “would-be” bound states become barely unbound, turning into low-energy, long-lived resonances. In appendix A, we give some numerical evidence that it is possible to tune the parameters of the two-body interaction of eq. (5.1) to create an arbitrary number of nearly degenerate low-energy resonances.

Qualitatively, one expects that the many different resonant states will behave as if they were the different species in a multi-species fluid. However, since these states are resonant states and not bound states, they eventually decay back into the fundamental particles, meaning this really is an interacting single-species gas rather than a multi-species gas. Furthermore, since the fundamental particles are absolutely stable, this system describes a stable fluid.

For the system to be of a single species, it is critical that the system does not have any three- or higher-body bound states. Given the singular nature of delta functions, one might worry that the Hamiltonian for three-body or higher-body Hilbert spaces might be unbounded from below, yielding arbitrarily deeply bound states. By regulating the delta functions and treating them as finite width wells, it should become readily apparent that this will not occur in the zero width limit with fixed resonance positions. Yet, it is not immediately apparent whether or not the system, as given, supports three- or higher-body bound states. To ensure that such states are excluded from our system we also impose a three-body repulsive potential. We choose the three-body interaction  $V_3(\vec{r}_1, \vec{r}_2, \vec{r}_3)$  to be

$$\begin{aligned}
 V_3(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= V_3 \Theta(R - \max[l_1, l_2, l_3]), \\
 l_1 &= |\vec{r}_1 - R_{CM}|, \\
 l_2 &= |\vec{r}_2 - R_{CM}|, \\
 l_3 &= |\vec{r}_3 - R_{CM}|, \\
 R_{CM} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3}{m_1 + m_2 + m_3},
 \end{aligned}
 \tag{5.2}$$

where  $V_3$ , the strength of the three-body interaction, is a constant set to be larger than any other energy scale in the problem,  $\vec{r}_1$ ,  $\vec{r}_2$ , and  $\vec{r}_3$  are the position vectors of the three interacting particles,  $R$  is the range of the three-body interaction,  $R_{CM}$  is the location of the center of mass, and  $l_1$ ,  $l_2$ , and  $l_3$  are the distances from the center of mass to the location of each particle. The range of the three-body interaction range  $R$  is chosen to be larger than the range of the two-body interaction  $L$ . This interaction forces the interaction between the fundamental particles and any resonant state to be that of hard sphere scattering. Once a two-particle resonance is formed, the three-body potential above prevents the resonance from being disturbed by interactions with other particles and prevents the formation of three-particle resonant states.

## 5.2 Constructing a bound on the entropy

The calculation of the entropy of a strongly coupled many-body system can be quite difficult. Instead we use a variational argument which shows that entropy of the entire system for a gas of many particles interacting through eq. (5.1) can be bounded from below. In the next subsection, we will choose a variational ansatz for which the bound is calculable and show that the lower bound of the entropy can be made arbitrarily large.

Since the fluid under consideration has a finite temperature, we can work in the canonical ensemble. Recall that in this ensemble, with natural units ( $k_B = 1$ ), the entropy is given by

$$S = \frac{E}{T} + \log(Z). \quad (5.3)$$

where  $E$  is the energy of the system,  $T$  is the temperature, and  $Z$  is the partition function. By increasing the step height in eq. (5.1), we can tune the system to have only resonant scattering states, and no two-body bound states. Similarly by choosing the strength of the repulsive three-body potential in eq. (5.2) large enough, we can ensure that there are no three- or higher-body bound states. This means that all of the possible configurations of the fluid must have positive energy. Therefore, the entropy is bounded by

$$S \geq \log(Z). \quad (5.4)$$

Just as with the entropy, the partition function is difficult to calculate directly, but the partition function is also bounded from below.

Recall that in the canonical ensemble the partition function is given by

$$Z = \text{Tr}(\exp[-\beta\hat{H}]), \quad (5.5)$$

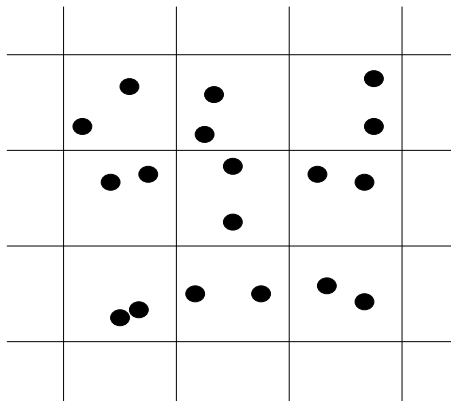
where  $\hat{H}$  is the Hamiltonian operator for the system and  $\beta$  is the inverse temperature. In order to compute the partition function, one typically needs to use a complete basis for the Hilbert space of the system. Since the Hamiltonian is Hermitian, the operator  $\exp[-\beta\hat{H}]$  is positive semi-definite. This implies that the partial trace over any arbitrary subspace of the Hilbert space gives a lower bound on  $Z$ , termed  $Z_{\text{sub}}$ . Choosing such a subspace amounts to choosing a variational ansatz for the class of configurations of the fluid: a calculation of the partition function within the variational ansatz is equivalent to the partition function of some subspace of the complete Hilbert space. Furthermore, the relation of the partition functions holds for the logarithm of the partition function as well,

$$\log(Z) \geq \log(Z_{\text{sub}}). \quad (5.6)$$

Combining eqs. (5.4) and (5.6) yields

$$S \geq \log(Z_{\text{sub}}). \quad (5.7)$$

This shows that the entropy of the entire system is bounded from below by  $\log(Z_{\text{sub}})$ . By working with a variational ansatz for which the partition function  $Z_{\text{sub}}$  is calculable, we can compute a lower bound on the entropy of the fluid.



**Figure 1:** As a variational ansatz, we picture the fluid’s volume to be divided into cells with exactly two particles in each cell.

### 5.3 Calculating the partition function

In this subsection we choose a variational ansatz for the system for which the calculation of the lower bound for the entropy is tractable. The particular configuration of the system that we consider is picked entirely for computation ease and is a highly unlikely one. This merely ensures that the true entropy may be well above our computed lower bound.

Consider dividing the volume occupied by the fluid into cells. For our variational ansatz, we will choose to have exactly two particles in each cell. The total wave function for this ansatz can be constructed out of the wave function for each cell as:

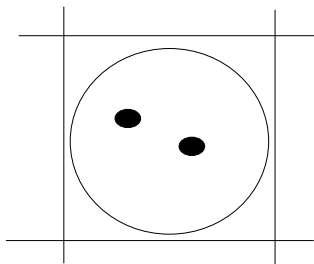
$$\Psi_{\text{total}}(r_1^{\vec{r}}, r_2^{\vec{r}}, \dots) = \hat{S} \prod_{\text{cells } i} \Psi_i(r_{2i-1}, r_{2i}) \tag{5.8}$$

where  $\Psi_{\text{total}}$ , the wave function of the entire fluid, is a function of the position of every fundamental particle in the fluid,  $\hat{S}$  is an operator which symmetrizes the wave function under the exchange of any two particles to impose the exchange symmetry of bosons, and  $\Psi_i$  is the (two particle) wave function of each individual cell, and they are summed over all of the cells. An illustration of the cell decomposition of the fluid is given in figure 1. To make the computation of the entropy easier, we further restrict the configurations so that wave function for each cell has the relative coordinate and center of mass coordinate completely uncorrelated. With this choice, the wave function for a cell can be written as

$$\Psi_{\text{cell}}(\vec{r}, \vec{R}) = \Psi_{\text{rel}}(\vec{r})\Psi_{\text{CM}}(\vec{R}), \tag{5.9}$$

where  $\vec{r}$  is the relative coordinate,  $\vec{R}$  is the center of mass coordinate,  $\Psi_{\text{rel}}$  is the wave function associated with the relative coordinate, and  $\Psi_{\text{CM}}$  is the wave function associated with the center of mass. Our ansatz is subject to one further condition: namely, the





**Figure 2:** A close up view of one particular cell with the drawn circle representing the constraints on the particles wave function imposed by the boundary conditions.

following (Dirichlet) boundary conditions:

$$\begin{aligned} \Psi_{\text{rel}}(\vec{r})|_{r \geq r_{\text{max}}} &= 0, \\ \Psi_{\text{CM}}(\vec{R})|_{R \geq R_{\text{max}}} &= 0, \end{aligned} \tag{5.10}$$

where  $r_{\text{max}}$  and  $R_{\text{max}}$  are the maximum relative coordinate and center of mass coordinate, respectively, that is allowed by a given cell. We take  $r_{\text{max}} > L$ , so that the maximum relative coordinate is beyond the range of the two-body interaction. These boundary conditions ensure that for this particular ansatz the fundamental particles only interact within a given cell, and that each cell is isolated from all other cells. This isolation implies that the two-body interaction plus the boundary conditions give the dominant contribution to the partition function within the subspace that we are considering. A pictorial view of the constraints of the boundary conditions can be seen in figure 2. This highly restrictive ansatz is certainly an unlikely configuration of the fluid, but it is a valid variational ansatz; such configurations are present in the complete Hilbert space.

Having chosen an ansatz for the wave function of the fluid, we can compute the corresponding partition function. The arguments of the preceding subsection showed that since the fluid that we consider has only positive energy states, the entropy of the entire system will be larger than the logarithm of the partition function calculated in this ansatz. We have isolated each cell by imposing boundary conditions, and it is sufficient to calculate the partition function of only one cell to exhibit the bound. Since each cell is identical, the total entropy within the ansatz is the entropy of one cell times the number of cells. Accordingly the entropy density of the fluid is bounded by:

$$s \geq \frac{n}{2} S_{\text{cell}} \tag{5.11}$$

where  $n$  is the total density (implying that  $n/2$  is the density of cells, and the factor of  $\frac{1}{2}$  is due to our choice of two particles per cell).

In order to show that the entropy density of the fluid is arbitrarily large, we only have to show that the logarithm of the partition function  $\log(Z_{\text{sub}})$  for one particular cell in

the fluid can be made arbitrarily large. To calculate the partition function, the energies of the states within each cell are needed. Since the two-body interaction has a finite range, the relative coordinate wave function within the cell has two different forms: one within the range of the interaction,  $\Psi_{\text{in}}$ , and one beyond the range of the interaction,  $\Psi_{\text{out}}$ . The outer wave function is that of a free state restricted by the boundary conditions, and can be written as

$$\Psi_{\text{out}}(r) = A \sin(k(r_{\text{max}} - r)), \quad (5.12)$$

where  $A$  is a normalization factor,  $k$  is the momentum of the state such that  $k = \sqrt{2\mu E}$  with  $\mu$  as the reduced mass, and  $E$  is the energy of the state. The momentum, and thereby the energy, of the quantum states within the cell can be calculated by matching the logarithmic derivative at the boundary between the two wave functions. The matching leads to the equation

$$\left. \frac{\Psi'_{\text{in}}(r)}{\Psi_{\text{in}}(r)} \right|_{r=L} = -k \cot(k(r_{\text{max}} - r))|_{r=L}. \quad (5.13)$$

The solutions of these equations give the energies of the states within each cell. Relating this condition to the two-body s-wave scattering phase shifts yields the condition:

$$kr_{\text{max}} = -\delta(k) + n\pi, \quad (5.14)$$

where  $n$  is an arbitrary integer. Since the phase shifts pass rapidly through  $\pi$  at each resonance, it should be apparent that there is one low-lying energy state within this ansatz for every resonance.

The parameters of the two-body interaction can be tuned in such a manner that all of the resonant states have nearly degenerate, arbitrarily low energies. If the resonance energies are fine-tuned to be very small compared to the temperature of the system, their contribution to the partition function is only slightly suppressed by a Boltzmann factor and each resonance contributes nearly unity to the  $Z_{\text{sub}}$ . From the resonant contributions it is easy to see that

$$\log(Z_{\text{sub}}) > \log(N) - E_H/T \quad (5.15)$$

where  $E_H$  is the energy of the highest-lying resonance. To the extent that  $E_H \gg T$  and  $N$  is large, the inequality is almost saturated; the logarithm of the partition function of the restricted system thus scales as  $\log(N)$ . We illustrate that this scaling can be realized by providing the results of numerical calculations in appendix A.

The bound established in the preceding subsection shows that the system's entropy density,  $s$ , is larger than  $\log(Z_{\text{sub}})$ . By increasing the number of resonant states while keeping  $E_H$  fixed, the lower bound on the entropy also increases. Since the number of resonant states in the two-body interaction can become arbitrarily large, so can the lower bound on the entropy density.

#### 5.4 Viscosity and stability

To complete the argument that the single-species fluid considered here can violate the class 2 variant of the KSS conjecture, we need to argue that the shear viscosity  $\eta$  does not grow

with the number of two-body resonant states,  $N$  (or, more precisely, grows slower than logarithmically). Furthermore, it is important to show the resulting fluid is stable in order to rule out variants of the conjecture of both classes 2a and 2b.

The shear viscosity is difficult to calculate for virtually any strongly-interacting system. For fluids for which the Boltzmann equation is applicable, there are simplifying arguments that allow one to calculate the shear viscosity [24]. However, due to presence of long-lived resonant states, the fluid described here does not satisfy the assumptions of the Boltzmann equation. Therefore, we know of no way to directly calculate the shear viscosity analytically.

Heuristically, the resonant states in the system described in this section can be thought of *approximately* as bound states. In section 4, we showed that the shear viscosity of a system of bound states need not scale uncontrollably with additional components to the fluid. Therefore, it is difficult to believe that the shear viscosity for the approximate bound states would scale vastly differently than that of a dilute many-component fluid. The actual difference between the shear viscosity of the two systems should depend on how well the bound state approximation holds, which depends on the resonant state lifetimes. We have constructed the resonant states of the fluid to have very long lifetimes. As a result, for the purposes of understanding the shear viscosity, the approximation that the resonant states can be considered bound states should be quite accurate. Therefore the shear viscosity of a fluid of long-lived resonant states should scale similarly to the viscosity of a fluid of bound states. Moreover, the shear viscosity of a fluid typically diverges only when it approaches either a non-interacting ideal gas, or behaves like the cold limit of a fluid without a defined melting temperature, such as glass. It is hard to see how a strongly interacting system, such as the one described in this paper, with a large number of long-lived resonant states should approach either one of these limits with the addition of resonant states. Therefore, the shear viscosity should remain finite as the entropy is made arbitrarily large, violating the  $\eta/s$  bound. While this is not a mathematically rigorous argument, it is very hard to see how it can fail.

In discussing shear viscosity, we approximated the system as though it contained bound states. However, at a fundamental level there are no bound states, and the fluid is still composed of only one species. If one wanted to compute  $\eta/s$  for this system numerically, for instance, the relevant degrees of freedom to simulate would be those of the fundamental particles together with their interactions, and *not* of the resonances. Since these fundamental particles are absolutely stable, by construction, the fluid is stable.

## 5.5 Summary of results on class 2

The preceding arguments show that the entropy, and therefore the entropy density increases with the number of resonant states. We have argued that although the calculation of the shear viscosity for the fluid we described is not tractable, there are strong heuristic reasons to believe that it will not diverge when one chooses parameters to force the entropy to diverge. To the extent one accepts these arguments, one must conclude that the ratio  $\eta/s$  can be made arbitrarily small by increasing the number of resonant states, violating the conjectured bound on  $\eta/s$ .

The number of resonant states needed to actually violate the bound could be extremely large, but the two-body interaction that has been discussed here can be tuned in such a manner as to produce an arbitrary number of resonant states. That is, there does not appear to be a limit inherent in the structure of quantum mechanics on the number of resonant states that can be constructed within a finite ranged potential.

We note that if a conjecture is false for stable fluids in some class of theories, it must be false for metastable fluids as well. As a result, the fluid that we have described in this section actually provides a counterexample to all theories of class 2, both for stable and metastable fluids.

## 6. Class 3

In the preceding two sections, we examined possible variants of the  $\eta/s$  bound for theories of classes 1 and 2 and argued that it is possible to construct systems in those contexts that violate the bound through a large Gibbs mixing entropy. As we noted in section 3.1, however, one might believe that the structure of quantum field theory (and specifically, the structure of “sensible” quantum field theories such as QCD) may rule out counterexamples based on the Gibbs mixing entropy. That is, the conjecture that  $\eta/s \geq 1/4\pi$  may be taken to apply only to systems that can be described by “sensible” quantum field theories. This form of the conjecture would be associated with class 3 and is similar to the variants proposed in refs. [2, 3]. In this section, we will review a counterexample to this class first presented in ref. [6], and give a more detailed discussion of some of the subtleties in that analysis. To conclude this section, we discuss a possible objection by Son [16] to the applicability of this counterexample to the KSS bound of class 3; we conclude that the issues raised by ref. [16] should not limit the applicability of the counterexample.

As we saw in section 3.1, a naive attempt to construct a system of light mesons with a very large number of different species by increasing the number of flavors  $N_f$  in QCD resulted in the ratio  $\eta/s$  scaling as  $N_f^2/\log N_f$ , implying that the bound held in the large  $N_f$  limit. Recall that this scaling of  $\eta/s$  was due to the fact that to preserve asymptotic freedom (and thus “sensibility”), as  $N_f$  was increased, the number of colors  $N_c$  also had to be increased proportionally to  $N_f$ . The bound then held because the cross section scaled as  $1/N_c^2$  in the large  $N_c \sim N_f$  limit. As it turns out, however, this result is not characteristic of *all* meson gases. In this section we review a counterexample first discussed in ref. [6] for theories of class 3 by considering a *heavy* meson gas.

### 6.1 A heavy meson gas

Consider a gas of heavy mesons. Each meson is made from a heavy quark and a light anti-quark. For the discussion that follows, we will assume that the gas is only composed from pseudoscalar heavy mesons, and will justify this assumption below. We can produce many heavy meson species by fixing the number of light quark flavors to some small value with one being adequate, and scaling the number of heavy quark flavors,  $N_f$ , to be large:  $N_f = e^{\xi^4}$ , where  $\xi$  is a dimensionless scaling parameter. As in section 4, this scaling is chosen to ensure that the Gibbs mixing entropy of the heavy meson gas scales as  $\xi^4$ , which

is what is necessary to drive the ratio  $\eta/s$  to zero. As before, to ensure asymptotic freedom, we must scale the number of colors,  $N_c$ , as we scale the number of heavy flavors, hence  $N_c = e^{\xi^4}$ . At this point, in the case of the light meson gas, the meson-meson cross section was seen to scale as  $1/N_c^2$ , and the resulting increase in the viscosity prevented a violation of the bound. However, the heavy meson cross section does not scale in the same way, and the same problem does not arise.

Recall that in the example of a nonrelativistic gas discussed in section 4, it is important to remain in the low-density, low temperature regime so that the calculation of both the entropy and the viscosity is tractable. In this regime, two-particle scattering is dominant, and the scattering is described by a Schrödinger equation for the relative wave function  $\psi$ ,

$$(-\nabla^2 + mV)\psi = mE\psi, \tag{6.1}$$

where  $m$  is the mass of each of the the interacting particle,  $V$  is the interaction potential, and  $E$  is the energy of scattering associated with the relative motion of the particles. The critical point is that the cross section depends only on the combinations  $mV$  and  $mE$ , but not on  $V$  or  $E$  separately.

In the nonrelativistic gas case of section 4,  $mV$  is scale independent by construction (since neither  $m$  nor  $V$  scale with  $\xi$ ), while  $mE$  scales like the temperature, since the typical energy of two-body scatterings within a gas is proportional to the temperature  $T$ . Therefore,  $mE \sim mT \sim mT_0\xi^{-2}$ , implying that classical two-particle, low-energy scattering is dominant (assuming that the density is sufficiently low, as it is with the scaling relations of eq. (4.1)). This implies that the cross section becomes scale independent in the large  $\xi$  limit. For the pion gas, by contrast, the interaction potential  $V$  scales as  $1/N_c$ , and  $N_c$  must be large to maintain asymptotic freedom in the large  $N_f$  limit. The mass of the pions is scale independent (as is the mass of all light mesons in large  $N_c$  limit), and thus in the pion gas  $mV$  scales as  $1/N_c$ , rather than being scale independent. Since  $N_c$  has the same scaling as  $N_f$ , the cross section becomes small in the large  $N_f$  limit, preventing the arguments given in section 4 from applying to the pion gas. As a result, the pion gas is not a counterexample to the KSS bound. It is now not hard to see how a heavy meson gas might evade these problems: the mass of the heavy mesons can be chosen to scale in such a fashion that  $mV$  remains scale independent.

However, if the heavy meson mass were to scale with  $\xi$  to keep the combination  $mV$  scale independent, the other important combination,  $mE$ , would no longer scale as  $\xi^{-2}$  as before unless the scaling of  $T$  were to be changed as well. One might be concerned that changing the way  $T$  scales may cause the system to no longer be in the regime of low temperature and low density. However, by fixing the scaling of  $mT$  to preserve the scaling of eq. (6.1), the low temperature and low density regime of eq. (4.1) is simultaneously maintained; by a judicious choice of scaling we can create a nonrelativistic heavy meson gas equivalent to the nonrelativistic system in section 4. The necessary scaling relations will be discussed below.

To understand the necessary scalings, let us begin by examining the heavy meson interaction. In the heavy meson gas, the interactions between heavy mesons at long distances

is mediated by the exchange of light mesons. That is, to leading order, the heavy meson interaction potential is a Yukawa potential,

$$V(r) \sim g^2 \frac{e^{-M_l r}}{r}, \tag{6.2}$$

where  $g$  is the heavy meson/light meson coupling, and  $M_l$  is the mass of a light meson. The mass of the light mesons is set by  $\Lambda_{\text{QCD}}$ , and we can choose  $\Lambda_{\text{QCD}}$ , and hence  $M_l$ , to be independent of  $\xi$ . By choosing  $M_l$  to be scale independent, we show that the range of the interaction becomes scale independent as well. The scale dependence of the potential strength  $V(r)$  is then simply given by  $g^2$ . In the large  $N_c$  limit, we expect that  $g \sim 1/N_c^{1/2}$ , and thus  $V \sim 1/N_c$ . We can now choose the heavy meson mass,  $M_h$  to scale as  $N_c$  so that  $M_h V$  remains scale independent, as desired. The mass of the heavy meson is dominated by the mass of the heavy quark. By scaling the heavy quark mass appropriately, the heavy meson mass can be fixed to scale as  $N_c$ . It is easy to see that choosing the heavy quark mass,  $m_h$ , to scale as  $m_h = m_{h_0} e^{\xi^4}$ , where  $m_{h_0}$  is the scale-independent portion of the heavy quark mass, will result in the correct scaling of the heavy meson mass,  $M_h = M_{h_0} e^{\xi^4}$ , where  $M_{h_0}$  is the scale independent portion of the heavy meson mass. These scaling relations ensure that the relevant quantity  $M_h V$  remains scale-independent. Note that while this simple argument was given in terms of a meson-exchange picture, valid at long distance, the scaling arguments hold quite generally.

In the nonrelativistic gas, we saw that  $mT$  scales like  $1/\xi^2$ . To be consistent with this in the case of the heavy meson gas, we can set  $T \sim 1/(M_h \xi^2) = T_0 e^{-\xi^4}/\xi^2$  with  $T_0$  a  $\xi$ -independent constant. This scaling relation preserves the scaling of  $M_h T \sim \xi^{-2}$  by construction, and therefore the relations of eq. (4.1) can be satisfied by choosing the density,  $n$ , to scale as before,  $n \sim n_0 \xi^{-4}$ . Finally, the light quark mass,  $m_l$ , should be scale independent, as is the light meson mass. To summarize, the parameters of the heavy meson gas must scale as follows:

$$\begin{aligned} N_c &= e^{\xi^4} & N_f &= e^{\xi^4} & m_h &= m_{h_0} e^{\xi^4} & m_l &\sim m_{l_0} \\ \Lambda_{\text{QCD}} &= \Lambda_{\text{QCD}_0} & n &= n_0 \xi^{-4} & T &= T_0 \frac{e^{-\xi^4}}{\xi^2}. \end{aligned} \tag{6.3}$$

With these scaling relations, we can simply repeat the argument given in section 4, and conclude that the viscosity  $\eta$  scales as  $\xi^3$ , while the entropy density  $s$  scales as  $\xi^4$ . This implies that the ratio  $\eta/s$  scales as  $\xi^{-1}$ . By taking  $\xi$  to infinity, we can thus drive the ratio to zero for a system described by a “sensible” (that is, asymptotically free) quantum field theory, violating the KSS conjecture for a system described by a theory of class 3.

We now justify the assumption that the heavy meson gas is dominantly composed from spin-0 pseudoscalar mesons, as opposed to spin-1 vector mesons as  $\xi \rightarrow \infty$ . Because of the scaling relations chosen in eq. (6.3), the heavy meson gas is clearly in the heavy quark limit. In this limit, the pseudoscalar,  $H$ , and vector,  $H^*$ , heavy mesons are nearly degenerate. Therefore, one may naively expect both spin states to be present in the heavy meson gas. However, the two spin states have a typical mass splitting on the order of  $\frac{\Lambda_{\text{QCD}}^2}{m_h}$  [28]. From the parameter scaling relations in eq. (6.3), it is not hard to see that this mass splitting

scales like  $\sim e^{-\xi^4} \Lambda_{\text{QCD}}^2/m_{h0}$ . We would expect a heavy meson gas to contain both the pseudoscalar and the vector form of the heavy mesons, with their populations determined by a Boltzmann distribution. Let us consider the ratio of the populations of the vector mesons to the pseudoscalar mesons. From the Boltzmann distribution, this ratio is given by

$$\frac{N_{\text{vec}}}{N_{\text{pseudo}}} \sim \frac{e^{-\beta M_{H^*}}}{e^{-\beta M_H}} = e^{-\beta(M_{H^*}-M_H)}, \quad (6.4)$$

where  $\beta$  is the inverse temperature. Using eq. (6.3), we see that

$$\frac{N_{\text{vec}}}{N_{\text{pseudo}}} \sim e^{-\xi^2 \frac{\Lambda_{\text{QCD}}^2}{m_{h0}}}, \quad (6.5)$$

which for large values of  $\xi$  reduces this ratio to zero. Therefore, at large  $\xi$  the gas is predominantly composed of pseudoscalar heavy mesons, and we are well justified in neglecting the heavy vector mesons.

## 6.2 Stability

As we have done with the other counterexamples, the stability of the fluid needs to be considered. The system that we have constructed is actually metastable, and thus is a counterexample to conjectures of class 3b. Because of the attractive nature of the potential between heavy mesons, there are several ways by which the heavy meson gas may decay. However, the decay time scales are perimetrically large, implying that the gas is metastable.

In this context, it is natural to consider whether the heavy meson gas might be susceptible to decay through the formation of tetraquarks or other multiple meson states. As is well known, as one approaches the limit of infinitely heavy quark masses (with light masses held fixed) bound states of two heavy mesons, tetraquarks, must exist [29, 30]. The reason for this is simple: the color Coulomb interaction between the two heavy quarks allows the formation of a tightly bound diquark to which the two light antiquarks then bind. An alternative argument is that there is an effective potential between the two heavy mesons, the long distance part of which is given by pion exchange which always has an attractive channel when one includes both vector and pseudoscalars. It is a general theorem of elementary quantum mechanics that any potential with an attractive region always has two-particle bound states in the limit that the reduced mass becomes large. Since we are considering the limit of arbitrarily high masses with our scaling rules, one might expect that bound tetraquarks will exist at large  $\xi$  and lead to metastability. However, this generic theorem that bound tetraquarks must exist for large enough heavy quark mass assumes a fixed number of colors  $N_c$ . The relevant combination is in fact  $m_h N_c^{-1}$  since  $\alpha_s \sim N_c$ . Thus, the relevant parameter is  $m_{h0} \sim m_h N_c^{-1}$ . If  $m_{h0}$  is small enough relative to  $\Lambda_{\text{QCD}}$ , the tetraquark will not bind. Thus, by a judicious choice of parameters one can always prevent metastability due to the formation of tetraquarks.

It is plausible that there exist values of  $m_{h0}$  small enough so that stable tetraquarks do not exist but hexaquarks do. Presumably by fixing  $m_{h0}$  to be smaller still, one can ensure that stable hexaquarks also do not exist. More generally, it should be possible to ensure

that any  $k$ -heavy meson bound states up to some fixed  $k$  are unbound. This means that to the extent the system is metastable, it has a very long lifetime: from the arguments of section 4.2 it is clear that  $\tau_{\text{meta}}/\tau_{\text{fl}} \sim \xi^{5(k-1)}$ , which for large  $\xi$  is very large indeed.

One might hope that by choosing  $m_{h_0}$  to be small enough, we could eliminate *all* possible modes of rearrangement, and thus obtain a stable fluid, as opposed to merely a *very* long-lived metastable configuration. However, this is not the case: the process in which  $N_c$  heavy mesons rearrange into a heavy baryon and a light anti-baryon is always possible. As we show in appendix B, it is energetically favorable for the system to rearrange itself into a heavy baryon and a light anti-baryon. The binding energy of a baryon made from heavy quarks scales as  $N_c m_h (N_c \alpha_s)^2$  (the binding energy of a light baryon is order  $\Lambda_{\text{QCD}}$  and is negligible), while the binding energy of a system of  $N_c$  heavy mesons scales as  $N_c \Lambda_{\text{QCD}}$ . If the heavy quark mass  $m_h$  is large enough, it is energetically favorable for  $N_c$  heavy mesons to rearrange themselves to form a heavy baryon and a light anti-baryon. As a result, the heavy meson gas cannot be absolutely stable.

However, the time scale for the decay of the gas through this process is parametrically extraordinarily long. Recall that the scaling relations for the heavy meson gas are chosen to keep it in a low-density regime, and that  $N_c$ -body interactions (i.e., collisions) are necessary to convert  $N_c$  heavy mesons into two baryons. Such interactions are very rare, and the frequency of such  $N_c$ -particle collisions decreases with the density of the gas. By the standard arguments used previously, the ratio of time scales if the metastability is due to this mechanism is astoundingly large, scaling as

$$\tau_{\text{meta}}/\tau_{\text{fl}} \sim \xi^{5(\exp[\xi^4]-1)}.$$

For quite modest values of  $\xi$  this is an exceptionally long time. It is not totally clear that this scaling is relevant since it may be that an instability due to clumping of some finite but large number of heavy mesons,  $k$ , might always occur before this process sets in. In any case, the lifetime of the metastable state can be shown to be extraordinarily large.

There is one more decay mechanism for the heavy meson gas which should be mentioned. Recall that, in section 3.2, we mentioned two general types of metastable decays. For most of this paper, we have been discussing the type where the fluid is locally unstable, but the time scale of the decay is long. However, the heavy meson gas may also be metastable in the sense of being locally stable but globally unstable. In the types of fluids with this sort of instability, the system will typically remain in the metastable state for extremely long periods of time, usually due to some potential barrier, until a large perturbation forces the system into the lower-energy stable configuration. It may be possible that the heavy meson gas is an example of this type of fluid, but since we are not violently perturbing the system externally nor is there an internal mechanism to do so, the time scale associated with such decays is very large, i.e. scaling exponentially with  $\xi$ . Therefore, this possibility does not alter our conclusions.

### 6.3 The interplay of metastability and the thermodynamic and hydrodynamic limits

Son [16] has raised an interesting and subtle issue regarding the interplay of metastability



and the thermodynamic and hydrodynamic limits for the heavy meson gas. In doing so, he argues that because of the peculiar nature of this interplay in the heavy meson system, it is unreasonable to expect the KSS bound to apply. If one accepts this argument, then the counterexample given in this section, while valid on its own terms, does not provide evidence against the validity of the bound for more normal systems. However, as discussed briefly in ref. [31], the issue raised in ref. [16] does not appear to remove the heavy meson system discussed above from the class of theories for which a sensible bound ought to apply. Thus, we believe that the conclusions drawn from the existence of this counterexample do not need to be altered due to the arguments raised in ref. [16]. In this subsection, we outline the issue raised in ref. [16] and discuss its resolution.

The entropy density of a gas becomes well-defined in the thermodynamic limit. This means that the size of a system is large enough to contain a sufficiently large number of particles of each of the possible particle species in the gas so that the entropy density becomes well-defined. Let us define  $V_t$  as the volume in which (on average) we have one meson of every species:

$$V_t \equiv L_t^3 \equiv \frac{N_f}{n} = \frac{\xi^4 e^{\xi^4}}{n_0}, \quad (6.6)$$

where the final form imposes the scaling rules from eq. (6.3). As defined above,  $V_t$  defines the characteristic volume scale that is associated with the thermodynamic limit.  $L_t^3$  is the characteristic ‘thermodynamic length scale’ introduced in ref. [16]. It should be clear that to be in the thermodynamic limit, the physical volume of the system must be much larger than this characteristic volume. Clearly, from the  $\xi$  scaling in eq. (6.6), *very* large systems are required to achieve the thermodynamic limit for  $s$ . We note in passing that other thermodynamic observables, such as energy density or pressure, approach their thermodynamic limit much more rapidly than the entropy density since the thermodynamic limit of the entropy density alone depends on every species being present in a large numbers; hence, other thermodynamic observables do not require exponentially large systems. In this respect, the heavy meson system studied in this section is very unusual.

The viscosity, on the other hand, becomes well-defined in the hydrodynamic limit. This requires that viscosity measurements be performed on a length scale  $L_h$  or larger, where  $L_h$  sets a lower bound on the scale for which fluid behavior is evident. For dilute systems, such as the one under consideration here,  $L_h$  is effectively the mean free path,  $l_{mf}$ . Using the scaling relations in eq. (6.3) one sees that  $L_h \sim \xi^4$ .

For common fluids such as water or nitroglycerin, the hydrodynamic length scale  $L_h$  is generally comparable to, or larger than, the thermodynamic length scale  $L_t$ . For typical dilute gases with one or a few species of particle,  $L_h \geq L_t$ , since the mean free path is much larger than the average interparticle spacing. The heavy meson gas considered here is quite unusual in that  $L_t \gg L_h$ . Because of this fact and the metastable nature of the fluid, one might think that the bound should not apply to such systems, as argued in ref. [16].

To see the issue, suppose we want to *measure*  $\eta/s$  for some system composed of the heavy meson gas. At first glance there is nothing associated with the metastable nature of the fluid to prevent one from doing this to very high accuracy (at sufficiently large  $\xi$ ). In order to approach the hydrodynamic limit for which  $\eta$  is well defined, one needs to measure

$\eta$  in a system (or a part of a system) which is large compared to hydrodynamic length scales. Since as shown above, the ratio of the life-time of the fluid to the mean collision time is a positive power law in  $\xi$  (or higher), one can measure the viscosity over a system much larger than  $L_h$  long before the system decays. Thus,  $\eta$  is essentially well-defined as a hydrodynamic quantity. Similarly,  $s$  is essentially well-defined thermodynamically. The issue of concern here is whether the fact that  $\eta$  is essentially well-defined hydrodynamically is sufficient for the KSS bound to apply.

One natural perspective is that it ought to be sufficient. If the bound is general, one might think that it ought to apply to any system in which  $\eta$  (a hydrodynamical quantity) is essentially well-defined hydrodynamically, and  $s$  (a thermodynamical quantity) is essentially well-defined thermodynamically. There is an alternative perspective [16], however. Since the bound relates  $s$  to  $\eta$ , it is not unnatural to suggest that it should only apply when  $\eta$  and  $s$  are both *simultaneously* well defined in the sense of being simultaneously measurable in the same system.

If one adopts the latter view, there is a potentially serious problem. While the fluid clearly lives long enough to measure  $\eta$  accurately over a hydrodynamic length scale, it is very likely that the system would decay before  $\eta$  could be measured over the exponentially larger thermodynamic length scale. Accordingly, ref. [16] argues that because  $\eta$  and  $s$  cannot be determined simultaneously in the heavy meson system, the system is not in the class of systems for which the bound is expected to apply. If this is true, then despite the fact that the heavy meson gas on its own can violate the inequality  $\eta/s \geq 1/4\pi$ , it does not undermine the possibility of the existence of a bound which applies to more ‘normal’ systems arising from underlying UV-complete field theories even if they are metastable.

*A priori*, it is difficult to assess which of the two perspectives is likely to be correct. The bound is conjectured rather than derived, and accordingly its underlying assumptions are unclear. Thus, one might worry that if the second perspective turns out to be correct, and that the bound only applies when  $\eta$  and  $s$  are both *simultaneously* well defined for the same system, then the heavy meson example would not serve as a counterexample. However, as we will show below, despite the argument of ref. [16] outlined above, this perspective does *not* invalidate the heavy meson counter example.

The key point is that while the argument was formulated in terms of lengths scales, the thermodynamic limit depends on *volumes*. Recall that in general, for a system to be in the thermodynamic limit, the volume of the system is required to be large enough so that repeated measurements of thermodynamic quantities produce the same results, and that intensive quantities should be independent of the volume and the *shape* of the system. In practice, for dilute systems this means that a system is effectively in the thermodynamic limit if *i.*) the system is large enough to contain a sufficiently large number of particles of each species, and *ii.*) the system is characteristically thicker than the thermal wavelength  $1/\sqrt{mT}$  for all particles and in all directions. The second condition basically says that quantum uncertainties in where particles are located are small compared to the size of the system. This condition on the thickness of the system is the *only* condition on the length scales of the system, as opposed to the volume. It is easy to see that for systems with the scaling laws in eq. (6.3) the thermal wavelength scales as  $\xi^1$ . Since the mean free

path is always larger than  $1/\sqrt{mT}$  for the heavy meson gas, condition *ii.*) is automatically satisfied for any system with a thickness of the order of the hydrodynamic length scale or larger, and the question of whether the thermodynamic limit is reached depends only on the volume of the system, and not on the shape.

Given this, the notion of a thermodynamic length scale is not really well-defined: it depends on the arbitrary choice of a particular shape for thermodynamic system. With this in mind, consider as a simple illustration a non-relativistic, single-species ideal gas of particles of mass  $m$  in equilibrium at density  $n$  and temperature  $T$ , contained in a rectangular box. Suppose furthermore that the box is highly asymmetrical — the dimensions are  $W \times W \times t$  with  $W \gg t$ —and that condition *ii.*) is satisfied:  $t \gg 1/\sqrt{mT}$ . Now, the condition for the system to approach the thermodynamic limit with  $s$  well-defined amounts to the condition that the volume times the density is much larger than unity. This is satisfied provided that  $W^2 n \gg 1/t$ .

Two observations are in order here. The first is simply that the thickness,  $t$ , need *not* be larger than  $n^{-1/3}$  in order for the system to be in the thermodynamic limit. Indeed by making  $W$  large enough it is possible to take  $t$  to be *much smaller* than the interparticle spacing, and still have a consistent thermodynamic result. This can be explicitly verified by very elementary calculations. The second is that the result holds regardless of whether the rectangular region is considered to be a physical box containing the fluid, or merely as a fiducial volume in a much larger system.

This simple result for a single component fluid is trivially generalized for a multi-component fluid such as the heavy meson gas considered in this section. For the heavy meson gas we can again consider a slab geometry  $W \times W \times t$ , and find that the system is in the thermodynamic limit so far as entropy density is concerned provided that

$$W^2 \gg \frac{N_f}{n t} = \frac{1}{n_0 t} \xi^4 e^{\xi^4} . \tag{6.7}$$

Recall at this stage that measurements of  $\eta$  necessarily have a preferred direction. One considers a fluid with a velocity gradient transverse to the direction of fluid flow;  $\eta$  is the ratio of the stress to the magnitude of this gradient. Suppose that one wishes to measure the viscosity of the heavy meson fluid in the slab considered above, and takes the direction of the gradient to be the short side of the slab (*i.e.* along the thickness  $t$ ). The viscosity is essentially well defined hydrodynamically, provided that *a.*)  $t$  is much larger than the typical hydrodynamic scale and *b.*) the characteristic time for momenta to propagate through the thickness  $t$  is much shorter than the decay time of the fluid. Repeated measurements of  $\eta$  will then yield the same result up to very small fluctuations, so that  $\eta$  is well-defined. Equation (6.7) implies that if we choose  $W$  large enough, we can always ensure that the system is *simultaneously* in the thermodynamic limit with essentially well-defined  $s$ . To ensure that this is true, it is sufficient to take  $W = a \xi^2 \exp(\xi^4/2)$  and  $t = b \xi^4$ , with  $a$  and  $b$  sufficiently large constants. With this construction, at large  $\xi$ , we have a system which violates the KSS bound with  $\eta$  and  $s$  determined simultaneously and each essentially well defined. Moreover, if the system we are considering is large enough,

then regardless of its shape, one can always find a fiducial volume for which  $\eta$  and  $s$  can be measured simultaneously.

The upshot of this is that the requirement that  $\eta$  and  $s$  be determined simultaneously in the same system in order for the KSS bound to apply does *not* rule out the heavy meson system as a counterexample. At this point, one might object that the slab geometry considered is not general. However, this does not undermine the counterexample. The bound is supposed to hold generally for all systems arising from a “sensible” quantum field theory, with  $\eta$  and  $s$  are essentially well-defined and measurable simultaneously. A system composed of the heavy meson gas in this slab-like geometry proves that this is not true. The fact that there exist other geometries in which the system decays before  $\eta$  is determined does not alter this.

#### 6.4 Class 3a

The previous counterexample does not rule out class 3a (or 3' which is a subclass of class 3a) since it involves a metastable fluid. However, we should note that these variants of the conjecture have quite limited domains of applicability. Recall from section 3.3 that variants of the conjecture of class 3a do not apply to ordinary fluids such as water since the quantum field theory underlying water, the standard model, allows nuclear reactions which can alter the makeup of the fluid, albeit over *very* long times. However, by hypothesis for class 3a, metastability with arbitrarily long lifetimes is assumed to be qualitatively different from stability. If this were not the case then our counterexample to class 3b would also eliminate 3a since the time scales can be made very long.

Moreover, the fact that class 3a is of such limited applicability reduces the amount of evidence available to support this variant of the conjecture. Recall that one of the strongest pieces of evidence for the KSS bound was empirical: everyday fluids like water appear to respect that bound; no known example violating it exists. However, this evidence does not apply to conjectures in class 3a, for the reasons noted above.

Finally we note that although much of the analysis in this paper concerns the distinction between metastable and stable fluids, it is quite reasonable to suppose that that this distinction is unlikely to be important. We take the view that, while it is logically possible for there to be a universal lower bound (of  $1/4\pi$ ) on  $\eta/s$  for *only* stable fluids described by sensible quantum field theories, it is very difficult to see why such a lower bound should not apply even approximately to metastable fluids with arbitrarily long lifetimes.

To summarize, in this section we have described a system that provides a counterexample to the variant of the  $\eta/s$  bound of class 3b. The counterexample system is described by a limit of QCD, a UV-complete quantum field theory, and is metastable with an arbitrarily long lifetime. This counterexample does not apply to variants of the conjecture of class 3a (and its subclass 3'), but this remaining variant has a very limited regime of validity, and has relatively little evidence in its support.

## 7. Summary and discussion

There have been a number of variants of the conjecture on a universal lower bound for

Variant	QGP	$He$	$H_2O$	$C_3H_5(NO_3)_3$
3a.	Y	N	N	N
3'.	Y	N	N	N

**Table 3:** Table showing if each remaining conjecture can be applied (at least approximately) to either the quark gluon plasma (QGP), liquid Helium ( $He$ ), water ( $H_2O$ ), liquid nitroglycerin ( $C_3H_5(NO_3)_3$ ); Y(es), N(o)

$\eta/s$  [1–3]. After classifying these variants based on their domains of applicability, we have critically examined several variants of the conjectures. The broadest conjecture that has been made is of class 1, that  $\eta/s \geq 1/4\pi$  for all fluids described by quantum mechanics. Of all of the forms of the conjecture, this one has the strongest empirical evidence in its support. However, there exist counterexamples to variants of class 1, as we discussed in section 4. The counterexample system constructed there is the prototype of the other counter-examples discussed in the paper: the ratio  $\eta/s$  is driven arbitrarily close to zero by tuning a system to have a very large entropy while the shear viscosity is held fixed.

In section 5, we discussed a counterexample to variants of class 2, that the bound holds for nonrelativistic systems of one species with spin-0 or spin-1/2. By choosing a peculiar interaction potential and tuning its parameters, we showed that the entropy of a gas can be made arbitrarily large while arguing that the shear viscosity can remain fixed, violating the bound. This form of the conjecture appears to have some limited empirical support, but the existence of a counterexample to it suggests that the problem with the bound is not limited to situations with an exponentially large number of species in the gas — a contrived but well-defined interaction potential can produce systems that will violate the bound.

Lastly, in section 6, we showed that the structure of “sensible” quantum field theories does not appear to forbid the construction of systems with the very large number of species necessary to construct the sort of counterexamples that we have discussed in the preceding sections. In particular, we exhibited a counter-example to conjectures of class 3b, giving an example of a metastable gas described by an asymptotically free limit of QCD that can violate the bound. We note that the subtle issue raised in ref. [16] does not appear to alter this. While class 3a is not ruled out, it has little evidence in its support, and applies to a very limited class of theories. To illustrate the limits on the applicability of variants of the conjecture of class 3a, we reproduce table 2 from section 3.3 above, with only the variants of the conjecture that have not been ruled out shown. It appears that only exotic fluids like the QGP remain as an example of fluids that might be constrained by this bound.

Finally, there is one class of theories which may respect the conjectured bound which we have not discussed thus far in this paper. Since the original conjecture was based on the AdS/CFT correspondence, the bound may only hold for field theories with gravity duals in five dimensions (for instance, conformal field theories). As mentioned in section 2, there is strong theoretical evidence that the bound is valid for this class of theories. Generally speaking, conformal field theories would be included in class 3, as they are UV-complete (i.e., “sensible”) quantum field theories. Though we have demonstrated that the bound

need not be respected for all theories of class 3, it is certainly possible that it holds for some subclass of UV-complete theories. Hence, one may argue that field theories with gravity duals are the “sensible” theories needed to maintain the bound.

We should recall at this stage that the restriction of the bound (if it is universal) to systems which can be described by UV-complete relativistic field theories is difficult to justify from first principles. After all, the speed of light  $c$  does not appear in the bound, as one might expect if the result is coming from the relativistic nature of the underlying field theory. Moreover, from a dynamical point of view, while it is clear that physics at the UV scale of the field theory might somehow affect low-energy observables such as  $\eta/s$ , it is very hard to understand how the KSS bound could naturally emerge.

To conclude, it appears that there are counter-examples to the forms of the conjecture which initially appear to be supported best; the remaining forms of the conjecture that there is a universal bound on  $\eta/s$  for some well-defined broad class of systems outside the original domain of conformal field theories have both limited applicability and little evidence in their support. If the bound is correct despite the apparent existence of the counter-examples described in this paper, it would have to be due to some physics beyond the frameworks of quantum mechanics and quantum field theory. It is conceivable that the bound has a justification related to quantum gravity [17] or string theory, but given our present level of understanding, it is very difficult to see exactly how this might come about.

## Acknowledgments

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## A. Numerical results from single species fluid model

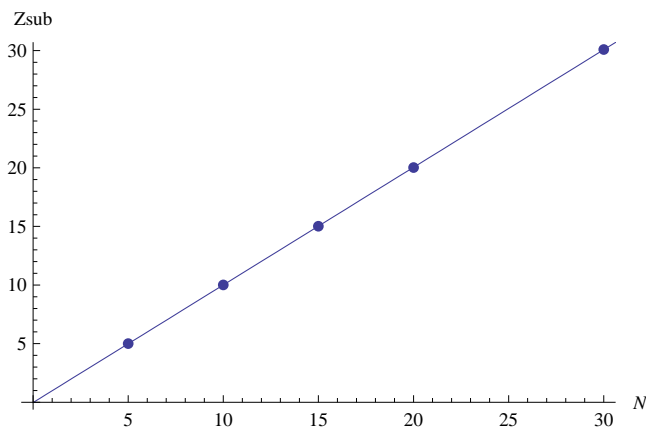
In this appendix, we present some numerical results in support of the argument that the partition function increases with the number of resonant states  $N$ . We neglect the effects of the states associated with the center of mass motion on the partition function and the effects of states associated with confining the wave functions in each cell, as these states are independent of  $N$ . The parameters of the potential eq. (5.1) were chosen as follows (in arbitrary units):

$$r_{\max} = 1.0001; \quad L = 1; \quad m = \frac{1}{2}; \quad \hbar = k_B = 1. \quad (\text{A.1})$$

Discussed in section 5, as  $N$  is increased, one has to tune  $b$  and  $V_0$  to produce narrow resonances. In table 4, we show the partition function and its logarithm calculated with increasing  $N$  and suitably tuned values of  $b$  and  $V_0$ , at fixed temperature. The values for  $b$  were chosen to ensure the resonant states were nearly degenerate, i.e. the larger  $b$ , the smaller the spread in energy of the resonant states. The values of  $V_0$  were chosen such that all states were barely resonant states and not bound states, while the temperature,  $T$ , was chosen large compared with the highest resonant state energy but smaller than the lowest-lying state associated with the artificial confinement to within a cell.

$N$	$b$	$V_0$	$T$	$Z_{\text{sub}}$	$\ln(Z_{\text{sub}})$
5	100	2,500.5	1600	5.06	1.62
10	200	10,001.8	1600	10.46	2.35
15	350	30,626.2	1600	15.46	2.74
20	480	57,601.9	1600	20.98	3.04
30	700	122,505	1600	30.09	3.40

**Table 4:** Numerical results showing the increase in the partition function  $Z_{\text{sub}}$  calculated using the variational ansatz.  $N$  is the number of resonant states;  $b$  is the strength of each delta function well in two-body interaction;  $V_0$  is the strength of energy plateau that creates resonant states in the delta function wells;  $T$  is the chosen temperature.



**Figure 3:** Graph of the calculated partition function and a linear best-fit to the data.

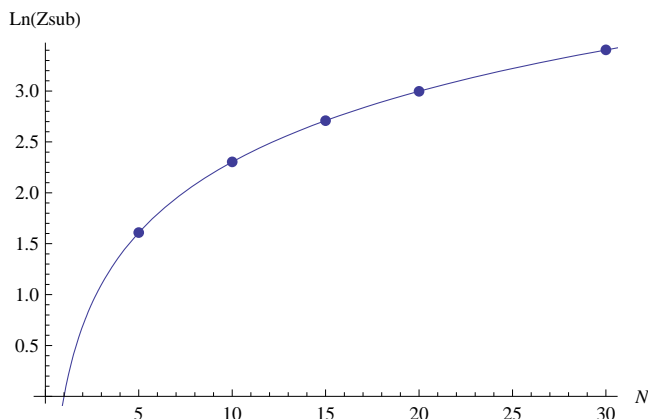
Note that the partition function and its logarithm scales with larger number of resonant states as expected by eq. (5.15). To further illustrate this, we plot the partition functions and their logarithms in figures 3 and 4, along with linear and logarithmic best-fit curves, respectively.

This numerical data supports the argument that by increasing the number of resonances in the potential of eq. (5.1), it is possible to increase the lower bound on the partition function of the system, and thereby increase the lower bound on the entropy.

## B. Metastability of heavy meson gas

As argued in section 6, the heavy meson gas that we discussed there is not susceptible to decay via tetraquark and higher-quark state formation. However, as we show in this appendix, one cannot tune the parameters of the heavy meson gas to prevent the formation of all bound states. It turns out to be energetically favorable for  $N_c$  heavy mesons to rearrange their quark content to form a heavy baryon and a light anti-baryon.

In order to show that the baryon configuration is more stable than the meson configuration, we must demonstrate that the binding energy is larger for the baryons than for



**Figure 4:** Graph of calculated logarithm of the partition function and a logarithmic best-fit to the data.

the mesons. The binding energy of a single heavy meson should be  $\sim \Lambda_{\text{QCD}}$ . However, a rearrangement into baryons needs  $N_c$  heavy mesons because the baryons must be color singlets; thus the total binding energy of the heavy mesons is  $N_c \Lambda_{\text{QCD}}$ . The binding energy of a light anti-baryon must be  $\sim \Lambda_{\text{QCD}}$ , as  $\Lambda_{\text{QCD}}$  is the only scale for the light quarks.

Next, we must determine the binding energy of the heavy baryon. As noted long ago by Witten [26], a baryon composed entirely of heavy quarks in the large  $N_c$  limit is described accurately in the Hartree approximation for nonrelativistic quarks interacting via a color Coulomb potential. We can calculate the binding energy relative for the heavy baryon parametrically via a virial theorem for this Hartree system. The single particle Hartree Hamiltonian  $H$  for the system with  $N_c$  heavy mesons has a kinetic term and a potential term:

$$\hat{H} = \hat{T} + \hat{V}. \tag{B.1}$$

The kinetic term,  $\hat{T}$ , can be expressed in the typical manner in terms of the Laplacian,

$$\hat{T} = \frac{-1}{2m_h} \nabla^2, \tag{B.2}$$

where  $m$  is the mass of the heavy quark. The potential term,  $V(r)$ , can be written using a mean field approximation. The leading order one-gluon potential that binds heavy quarks together has the form of a Coulombic potential, so in the mean field approximation, the potential term can be written as

$$\hat{V} = N_c \alpha_s \int \frac{\rho(r')}{|r - r'|} d^3 r', \tag{B.3}$$

where  $\alpha_s$ , the strong coupling constant, has been factored outside the integral, and  $\rho(r')$  is the particle density for one of the heavy quarks; the external factor is technically  $N_c - 1$  and indicates that each of the remaining quarks contributes. We will denote the exact single particle ground-state wave function of the Hamiltonian of eq. (B.1) by  $\Psi(r)$ .



In order to parameterize the energy of the ground state, instead of  $\Psi(r)$ , we choose  $\varphi(\lambda r)$  as a variational ansatz, with  $\lambda$  is the variational parameter. If we choose the form of  $\varphi(\lambda r)$  such that it happens to reproduce the form of the exact Hartree solution, the variational equations with respect to  $\lambda$  will yield an exact relation, i.e. a viral theorem. The ground state energy can now be determined by minimizing the Hamiltonian with regards to  $\lambda$ . In order to perform this minimization, we must first determine how the kinetic and potential terms scale with  $\lambda$ . Using the change of variable  $R \equiv \lambda r$  it is easy to see that the kinetic term must scale as

$$T(\lambda) \equiv \langle \varphi(\lambda r) | \hat{T} | \varphi(\lambda r) \rangle = \lambda^{-2} T(1) \equiv \frac{T_0}{m_h \lambda^2}, \quad (\text{B.4})$$

we have factored out  $1/m_h$  so that  $T_0$  is independent of  $\lambda$ ,  $m_h$ ,  $\alpha_s$ , and  $N_c$ .

To find the  $\lambda$ -scaling of the potential energy term, we first note that the single particle density can be written in terms of the wave function  $\Psi(r)$  as

$$\rho(r) = \Psi^*(r)\Psi(r). \quad (\text{B.5})$$

When we consider the scaling parameter, the density can be written as,

$$\rho(r) = \lambda^3 \varphi^*(R)\varphi(R) = \lambda^3 \rho(R) \quad (\text{B.6})$$

where  $R = \lambda r$  once again and the factor of  $\lambda^3$  comes from the normalization of the variational ansatz. With this expression for  $\rho(r)$ , the scaling of the potential is given by

$$\begin{aligned} V(\lambda) &\equiv \langle \varphi(\lambda r) | \hat{V} | \varphi(\lambda r) \rangle \\ &= N_c \alpha_s \int \frac{\rho(R')}{|\lambda r - R'|} d^3 R' \\ &= N_c \alpha_s \frac{1}{\lambda} \int \frac{\rho(r')}{|r - r'|} d^3 R' \equiv \lambda^{-1} N_c \alpha_s V_0, \end{aligned} \quad (\text{B.7})$$

where we first did a change of variables from  $R'$  to  $r' = \lambda^{-1} R'$ , and then factored  $\lambda$  out of the integral, leaving the factor  $V_0$  independent of  $\lambda$ ,  $m_h$ ,  $N_c$ , and  $\alpha_s$ . This has the effect of explicitly showing the  $\lambda$  scaling of the potential energy.

Using the above results, the Hamiltonian now takes a form where the  $\lambda$  scaling is fully explicit:

$$H(\lambda) = \frac{T_0}{m_h \lambda^2} + \frac{N_c \alpha_s}{\lambda} V_0. \quad (\text{B.8})$$

It is now easy to minimize this equation with respect to  $\lambda$ , and the variational estimate of the ground state turns out to be

$$E_0 = -m_h (N_c \alpha_s)^2 \frac{V_0^2}{4T_0}, \quad (\text{B.9})$$

where  $-E_0$  is the binding energy for one quark.  $E_0$  is expected to be negative, indicating a bound state; the binding energy of the heavy baryon is  $BE = -N_c m_h (N_c \alpha_s)^2 V_0^2 / (8T_0)$ .

Since by construction both  $T_0$  and  $V_0$  are factors depending only on the form of the variational wave function and independent of  $\lambda$ ,  $m_h$ ,  $N_c$ , and  $\alpha_s$  one has the following scaling of the binding energy with the parameters of the problem

$$BE(\text{heavy baryon}) \sim -N_c m_H (N_c \alpha_s)^2. \quad (\text{B.10})$$

Note that  $N_c \alpha_s$ , the square of the 't Hooft coupling is independent of  $N_c$ .

At this point we observe that before the rearrangement, the heavy mesons had a binding energy of  $N_c \Lambda_{\text{QCD}}$ , while after the rearrangement, the heavy baryon has a binding energy of  $N_c m_h (N_c \alpha_s)^2$ , while the light anti-baryon has a characteristic binding energy of  $\Lambda_{\text{QCD}}$ , which is negligible by comparison. Since the 't Hooft coupling constant scales like  $\sim 1/\log^2(m_h/\Lambda_{\text{QCD}})$ , the binding energy for the heavy baryon will always be perimetrically larger than for the  $N_c$  heavy mesons (which also scales as  $N_c \Lambda_{\text{QCD}}$ ) for a large enough value of the heavy quark mass. Therefore, this rearrangement of quarks is always energetically favorable, and thus the heavy meson gas is metastable relative to this rearrangement.

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